AD-A260 487

RESEARCH CENTER

COLLEGE OF ENGINEERING

UNIVERSITY OF PUERTO RICO

MAYAGUEZ CAMPUS

DTIC
S ELECTE
FEB 18 1993
B

Joint Flexibility Effects on the Dynamic Response of Structures - Part I : Deterministic Analysis

Final Report

by

Luis E. Suarez and Enrique E. Matheu

December 1992

*

prepared for

U.S. Army Research Office

Grant Number 28374-EG-H

University of Puerto Rico - Mayagüez

93-03134

REPORT DOCUMENTATION PAGE

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE December 1992	3. REPORT TYPE AND DATES COVERED Final Report

4. TITLE AND SUBTITLE

Joint Flexibility Effects on the Dynamic Response of Structures- Part I: Deterministic Analysis

5. FUNDING NUMBERS

G 28374-EG-H

6. AUTHOR(S)

Luis E. Suarez and Enrique E. Matheu

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

University of Puerto Rico- Mayaguez
College of Engineering
P.O. Box 5000
Mayaguez, PR 00681-5000

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

The objective of this study is to investigate the effect of the flexible connections on the dynamic behavior of structures. To take into account the uncertainties in their values, the stiffness of the connections are regarded as random variables. In the first part of the report the problem is studied from a deterministic point of view in order to develop the analytical models and expressions required for a stochastic analysis. A bibliographic review of the deterministic analysis of structures with non-rigid joints and analysis of structures with random parameters is presented. A finite element model that includes the effect of the flexibility and finite size of the connections is formulated. By using a variational approach it is shown that the mass matrix is also affected by the flexible connections A sensitivity analysis is carried out to assess how the natural frequencies and modal shapes are affected by the variation in the stiffness of the joints. Closed from expressions to calculate the derivatives of the eigenvalues and eigenvectors are provided.

14. SUBJECT TERMS

Flexible Connections - Finite ELement Models - Sensitivity Analysis-Perturbation Methods

15. NUMBER OF PAGES

94

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

DISCLAIMER

The view, opinions, and/or findings contained in this report are those of the authors and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

A-1

SCIENTIFIC PERSONNEL

The study presented in this report was carried out under the direction of the principal investigator, Dr. Luis E. Suarez, associate professor in the Department of General Engineering at the University of Puerto Rico-Mayaguez. He was assisted by Mr. Enrique E. Matheu, graduate student in the Department of Civil Engineering. Mr. Matheu completed his Master of Science degree in June 1992. He is now pursuing a Ph.D. degree in Engineering Mechanics at Virginia Polytechnic Institute. An undergraduate student, Mr. Jose Perez Suarez, also participated in the project. Mr. Perez completed his B.Sc. degree in June 1992 and then enrolled in the Master of Science in Civil Engineering program in UPR-Mayaguez.

Contents

1

List of Figures

Chapter 1

Introduction

1.1 Overview

The importance of the joint flexibility effects on the behavior of structural frameworks has been recognized for many years. This problem is relevant to diverse areas of structural engineering. In civil engineering for example, the commonly used methods of analysis and design of steel structures are based on the assumption that the member connections behave as pinned or perfectly rigid joints. However, early experimental studies on steel frames revealed that few real connections behave according to this assumption, and thus extensive research has been performed to characterize the behavior of flexible riveted and bolted beam-to-column connections. Besides the conventional steel building frames there are other civil engineering structures in which the flexibility of the joints can be important. For example, in steel bridges the degrading of the joint connections, besides other undesirable side effects, may significantly alter the dynamic characteristics of the structure. In mechanical engineering, it is important to assess the influence of body connection flexibility in vehicle structures which consist of irregular and complicated members connected by overlapping sheet metals fastened by spot welds. This is also the case in piping systems, in which in order to calculate the correct bending moment acting at the joints it is necessary to take into account the flexible behavior of the connections.

In view of the above, it is important to have available a rational and effective methodology to take into account the effect of the joint flexibility in both the dynamic properties and the overall response of the structure. The goal of the present study is first to examine the modeling of structures with flexible connections by finite element methods. Secondly, the effect of the joint flexibility on the natural frequencies, vibration modes and on diverse response quantities will be examined. The results of this research project should contribute to the better understanding of the behavior of structures with flexible connections. The study can also be very useful for the safe and economical design of these complex structural systems. The modeling of structures with partially rigid connections is a complex and challenging problem due to many factors that will be discussed in the sequel.

Due to the many factors and elements contributing to their behavior, the stiffness and other properties of the connections are usually known only with some degree of certainty. For example, the localized damping characteristics of a structural joint are the result of a complex combination

5

of several physical phenomena. Moreover, the local damping mechanisms can vary significantly in different parts of the structure, thus making their evaluation very difficult. In those cases in which the value of these structural parameters can be experimentally determined, the measured quantities should be regarded as the mean value of a set of experimental observations. Indeed, the set of observed values represents a distribution of the structural properties, with the spread in the distribution being a function of the uncertainties due to measurement inaccuracies or system complexities.

The best way to quantify through a rational approach the effects of the uncertainties in the parameters of the structural system is to use probabilistic methods, so that these structural parameters can be considered as random variables. The uncertainties can be mathematically modeled either as timevariant uncertainties or time-invariant uncertainties. The model of systems with parameters that vary randomly in time are very complicated to analyze since the response of the system is statistically correlated to these parameters. In general, it is very difficult if not impossible, to quantify the probability structure of the uncertain parameters. Therefore, it is common to use certain simplifying assumptions such as Gaussian density functions, stationarity, ergodicity or white noise characteristics. In the second model the system has parameters which vary randomly in space only, and the expressions "random field or stochastic fields" are used to denote these stochastic processes. If the uncertainties are limited to localized points of the system (e.g.joints), then the random parameters are represented not by a continuous random field but by a vector of random variables. This latter case is more amenable to analytical treatment, and the present research is focused on this type of uncertainties.

1.2   Literature Survey

1.2.1   Structures with Flexible Connections:

A review of the previous research work on this subject is presented in the sequel. For the organization of the presentation, the studies are divided into static and dynamic analyses. It must be pointed out that most of the reported research work on flexible connections is concerned with local deformation and stress analysis of the joints themselves [1, 2, 3]. Here, only those studies that deal with the overall effect of the flexible joints on the behavior of the structure will be surveyed.

1.2.1.1   Static Analysis

Most of the research on structures with flexible connections has been limited to static analysis. During the early sixties, the first methods to incorporate partially rigid connections to the traditional matrix analysis procedures were introduced. Monforton and Wu [4] derived relationships in matrix form between forces and displacements at the ends of members with elastically restrained ends. To calculate the stiffness matrix of a member with elastic restraints they proposed to modify the stiffness matrix of a member with rigid connections by multiplying it by a correction matrix. The elements of the correction matrix are a function of two parameters referred to as

6

the "fixity factors" of the member.

Romstad and Subramanian [5] presented another procedure to evaluate the partial connection rigidity in the design process using modified element stiffness matrices, putting special emphasis on stability analysis. Significant changes in the buckling capacity of the structures examined were reported after incorporating the connection flexibility effects. To take into account the nonlinear connection behavior due to local yielding, they used a bilinear model for the moment-rotation curves.

A linear static analysis of frameworks with elastic connectio·· was presented by Lightfoot and Le Messurier [6]. The stiffness matrix for a prismatic member with elastic spring supports at both ends was obtained in a closed form. A correction matrix amending the perfectly rigid condition was employed to obtain the stiffness matrix for two types of structural members (grids and plane frames). The corrected stiffness matrix was defined in terms of fixity factors for all the possible degrees of freedom of the two models.

Frye and Morris [7] also presented a procedure to incorporate the effects of flexible connections by modifying the stiffness matrix of plane frames. They proposed a method for expressing the moment-rotation behavior of all connections of a given type in terms of a single standardized function. A generalization of this procedure to three- dimensional steel frames was introduced by Ang and Morris [8].

A method to evaluate the drifts and moments of flexibly connected multi-story frames was presented by Arbabi [9]. The method, which is very simple to apply, gives exact results for symmetric single-bay frames subjected to static lateral loads.

Chen and Lui [10, 11] introduced a method for the nonlinear static analysis of plane framed structures with flexible steel beam-to-column connections. To account for the flexible connections, the stiffness matrix of a frame element is first modified by adding rotational springs at the ends of the member. The rotational degrees of freedom of the original member are then statically condensed. The force- displacements relationships are written in incremental form to incorporate the nonlinear behavior.

The nonlinear connection behavior as well as the geometric nonlinearity of the frame members were investigated in a study by Dhillon and Abdel-Majid [12]. A polynomial model was used to describe in a standardized form the moment-rotation curves of several types of connections. The secant stiffₙ ₛs coefficients of the beam-to-column connections were added to the stiffness and geometric matrices of the frame member using the same technique developed by Monforton and Wu [4]. Once they obtained the overall nonlinear stiffness of the frame member, the incremental equilibrium equations were solved through an iterative procedure. It was observed that the connection flexibility changes significantly the force distribution, causing a shift in bending moments from support to span. It was suggested that more economical structural design solutions could be found investigating the use of different types of connections with variable stiffness.

Many of the aforementioned studies present different but equivalent methods to incorporate the flexible connection effects into the common matrix analysis procedures thᵣ₀ugh the definition

of an appropriate stiffness matrix. These methods can be found nowadays in some textbooks on advanced structural analysis [13].

1.2.1.2 Dynamic Analysis

As mentioned before, most of the reported studies on structures with partial connection rigidity were limited to static analysis. Probably the first authors to study the dynamic behavior of these structures were Lionberger and Weaver [14] who examined the dynamic response of plane rectangular building frames. They adopted a bilinear relationship to approximate the actual moment-rotation curves. Axial deformations in beams and columns were considered negligible and the non-rigid connections were modeled by rotational springs at the ends of the members. The effect of finite connection size was taken into account by assuming that the portions of a frame element common to the intersections of beams and columns are rigid and introducing small rigid bars at each member end. The stiffness matrix of a typical member with these characteristics was developed. Damping in the frame was modeled using viscous dashpots at each floor level, where the masses were assumed concentrated. The possibility of elasto-plastic behavior of the columns was taken into account by inserting rigid-plastic connections at the end of column members. The method was used to analyze a multi-story frame subjected to an impulsive lateral load. The example presented demonstrated that the joint flexibility causes an appreciable increase in the overall dynamic response of the structure.

The vibrations of Timoshenko frames with flexible connections were studied by Yaghmai and Frohrib [15]. Using a continuous model, they derived the dynamic moment and shear slope-deflection equations for a Timoshenko beam element with flexible joints. By employing these equations along with the compatibility relations, they obtained the equations of motion for a one bay multi-story plane frame. The determinant of the dynamic matrix yielded the characteristic equation for this structure. Different beam-joint models were formulated and both the rotational and translational components of the joint deformation were taken into account. The joint flexibilities were assumed to obey a linear load-deflection relation. The effect of the flexibility and geometric size of the connections on beams with rotatory inertia and shear deformation were studied with numerical examples.

An analytical and experimental vibration analysis of framed structures was carried out by Kawashima and Fujimoto [16]. The energy-dissipating semi-rigid connections were idealized by rotatory springs and dashpots in parallel using the concept of equivalent viscous damping. The dynamic stiffness matrix was derived in an explicit form by means of a transfer matrix approach. The static stiffness, damping and mass matrices were obtained by expanding the elements of the dynamic stiffness matrix in power series with respect to the frequency of vibration and neglecting the terms higher than third order. A portal frame and an L-type frame with synthetic rubber connections were considered for the experimental investigation. The stiffness and damping factors of the connector were measured by a torsional free vibration test. The natural frequencies obtained from the test compared well against those predicted by the analytical formulation.

The dynamic response of flexibly connected frames, considering the nonlinear behavior of the connections, was studied by Shi and Atluri [17]. A Ramberg-Osgood model was used to approxi-

8

mate the actual moment-rotation characteristics for static and dynamic cases. A complementary energy approach was used to obtain the flexibility matrix of a frame member with rotational springs at its ends. Several numerical examples previously solved by other researchers were considered in order to check the accuracy of the model. The results compared well for both static and dynamic problems.

1.2.2 Uncertainties in Structural Parameters

The studies in which this problem was examined can be divided into two groups. The first approach is to consider the complete structural model as a deterministic system and to perform a sensitivity analysis to study the influence that a variation in a structural parameter has upon the response characteristics. The second approach is based on a probabilistic analysis of the system. and the effects of the uncertainties are evaluated by examining the probabilistic character of the response.

1.2.2.1 Sensitivity Analysis

A unified and complete presentation of the mathematical theory of structural design sensitivity analysis can be found in the monograph by Haug, Choi and Komkov [18], where the treatment of this subject is based on functional analysis and linear operator theory. The sensitivity of the static response, eigenvalue problem and dynamic response of finite-dimensional systems is analyzed. Both direct design differentiation and adjoint variable methods of design sensitivity analysis are presented. Infinite-dimensional problems are also treated in some detail.

An excellent survey of methods for calculating sensitivity derivatives for discrete structural systems was presented by Adelman and Haftka [19]. They examined most of the literature dealing with derivatives of static displacements and stresses, eigenproperties, transient response. and derivatives of optimum structural designs with respect to problem parameters.

With regard to the sensitivity of eigenvalue problems, the early work of Fox and Kapoor [20] deserves to be mentioned. They developed techniques for computing the first-order derivatives of eigenvalues and eigenvectors for a generalized eigenvalue problem with symmetric matrices. Two formulations for the calculation of the eigenvector derivatives were presented: the first method is based on the differentiation of the matrices in the eigenvalue problem and it leads to a set of simultaneous equations. One drawback of this approach is that the resulting set of equations is singular and it is necessary to perform some algebraic manipulation to overcome the singularity. This method leads to an expression for the rate of change of an eigenvector that involves only the eigenvalue and eigenvector under consideration, and thus a complete solution of the eigenproblem is not needed. The second method consists of expressing the eigenvector derivative as an expansion or series in terms of the eigenvectors. The coefficients of the series are obtained substituting the modal expansion into the differentiated form of the eigenvalue problem. This approach requires to know, in principle, the complete solution of the eigenproblem. However. it is possible to approximate the derivative by truncating the modal expansion, as it is often done in structural response calculations.

A general method for determining the sensitivity of structural response variables defined by

9

an eigenvalue problem or by a system of equations was presented by Romstad. Hutchinson and Runge [21]. They studied the effect of changes in the design parameters represented by a first-order perturbation of the structural matrices. Particular emphasis was placed on the sensitivity of the solution of the generalized eigenvalue problem. The eigenvalues and eigenvectors were defined in terms of power series of the perturbation parameter. The series are convergent for sufficiently small values of this parameter [22]. The expansions are then substituted into the eigenvalue problem and the equations for each order in the perturbation are separated. The method was applied to a simple structural stability problem where the changes in the first buckling load and mode shape caused by small modifications of the structural parameters were considered.

Nelson [23] presented a method to calculate the first-order derivatives of the eigenvectors for large eigenvalue problems. In this case, the modal expansion approach for expressing the eigenvector derivative may become prohibitively expensive. The proposed method also overcomes the singularity of the set of equations of the eigenvalue problem without destroying any bandness associated with the original matrices.

A survey of methods for sensitivity analysis of eigenvalue problems for non-hermitian matrices was presented by Murthy and Haftka [24]. They pointed out the importance of incorporating the second order derivatives into the analysis, since eigenvalues are usually nonlinear functions of the structural parameters and a second order approximation offers a much wider range of applicability than the first order expansion. The authors presented a comparative analysis of the various methods available for calculating the derivatives of the general eigenproblem and proposed some modifications to existing techniques.

1.2.2.2 Probabilistic Analysis

Randomness, defined as a lack of a pattern or regularity, is a common characteristic found in most engineering systems. There are two sources of randomness that can be observed in a physical system. One is a inherent irregularity in the phenomenon that makes impossible a deterministic description. The second source is the lack of complete knowledge about the characteristics of a given system and the processes that dictate its behavior. In principle, this level of uncertainty could be reduced by performing more observations of the system and improving their accuracy. However, sampling at all locations or at a relatively large number of points is usually impractical or uneconomical, and unavoidable measurements inaccuracies will tend to dilute the value of the information gathered. Therefore, the process of analysis and design must proceed on the basis of incomplete information about the medium. It is apparent from the preceding discussion that deterministic models can only be regarded as approximations to the actual physical reality, analogous to the linear models used as approximations to the actual nonlinear behavior of the systems.

The randomness in the response of a structural system can be induced by the excitation or by the system operators. Problems in which the input is random and the coefficient matrices are deterministic have been studied intensively for many years, especially in the field of signal processing and random vibrations. The case of systems with random operators, with random or deterministic input, is much more complicated and many of the analytical and computational

10

tools for the analysis of these systems are still at the development stage. In structural mechanics the random operators assume the form of algebraic or differential equations with random coefficients due to the spatial statistical variation of the geometric or material properties of the systems. The variation in space of the attributes of a physical system can be modeled by means of random variables or random processes. The random processes in space are referred to as random fields. The concepts and applications of random field theory can be found in Reference [25]. By representing a random process by its values at a discrete set of points in its domain of definition, the uncertainties in the distributed properties of the system can be modeled as random variables.

When the time response of the system is studied, the discretized form of the random field leads to a system of coupled ordinary differential equations with random coefficients. Although the formal study of the theory of random differential equations has been undertaken only recently, there is already a significant body of knowledge (see for example, References [26-29]). Many of these studies are devoted to fundamental mathematical concepts such as existence, uniqueness and stability of the solution, without focusing on methodology and practical applications.

An engineering-oriented treatment of the subject of differential equations with random coefficients can be found in Chapter 8 of a book by Soong [30]. Following a brief discussion of equations with constant coefficients and problems in which the coefficients have a completely erratic behavior in time (Gaussian white noise process), a more detailed analysis of problems with random coefficients that fall between these two extremes is presented. The perturbation technique was applied to obtain the approximate solution of a general vector differential equation with a random differential operator. The basic idea of the perturbation scheme is to expand the differential operator and the solution vector around their respective mean values via a Taylor series. By equating terms of the same order, the random system of equations is replaced by a theoretically infinite number of deterministic equations with random inputs. The random inputs are function of the randomness in the system since they depend on the solution of the lower order equations.

The final goal in the study of random differential equations is the determination of the family of distributions of the response variables. This goal, however, is extremely difficult and usually impossible to achieve. Besides the mathematical complexity of the problem, it requires considerable information about the joint probabilistic behavior of the random coefficients of the equations. However, quite often the only information available on the probabilistic structure of the coefficients is the first two moments, i.e., the mean and correlation functions. In many cases, for example for reliability analysis, engineering decisions are also based upon the first two moments of the solution process. It may be more efficient in this situation to try to determine directly the so called moment equations, that is, the set of differential equations satisfied by the moments of the solution process. It can be shown that the moment equations of a given order contain only moments of the same or lower order which implies that the moments can be solved successively. Soong [30] applied this approach to the study of the random one dimensional wave equation and to a system of random spring-mass oscillators. Another procedures to obtain the solution properties outlined in Soong's monograph are the Truncated Hierarchy Techniques used in statistical physics. These methods, however, are not very popular for engineering problems.

11

A powerful approach for the solution of random differential equations is the so-called Stochastic (or Probabilistic) Finite Element Method (SFEM). This term embodies several numerical techniques used to simultaneously discretize and solve random differential equations. In its prevailing form, the SFEM involves the use of deterministic finite element techniques coupled with the perturbation method to estimate the mean and variance of the response of the system. The most attractive features of the SFEM are its conceptual simplicity and easy of implementation into existing finite element computer codes. A report on the state-of-the-art in stochastic finite elements and random fields was presented by Vanmarcke et al. [31]. This paper provides a brief overview of the fundamentals of random field theory, the second order representation of homogeneous random fields and their digital simulation. The problem of the discretization of the coordinate space of a random field is also examined. It is indicated that the resulting discretized stochastic representation and consequently, the accuracy of the solution, depends strongly on the finite element mesh size. An example of the SFEM formulation for static analysis of a structure subjected to deterministic loads is presented. The problem of misfit of members in assembled structures and the case of structures with elastic foundations are briefly considered. An outline for the application of SFEM to dynamic problems, eigenvalue analysis or time-history analysis, is also presented.

Another excellent review on the application of finite element methods for the treatment of random differential equations can be found in the review paper by Benaroya and Rehak [32]. They described the use of both the perturbation method and the linear partial derivative approach to obtain approximate solutions for systems with coefficients which are stochastic processes in space. It was shown that both methodologies result in the same governing equations. In the linear partial derivative method the first and second order correction terms of the perturbation expansion are defined as partial derivatives with respect to the random parameters evaluated at the mean values. Benaroya and Rehak pointed out that the variance of the uncertainties must be small in order to obtain acceptable accuracy when only a few terms of the infinite expansion series are used. As with all perturbation methods, neglecting higher order terms can be justified only when there is a rapid rate of convergence of the series. The rate of convergence is controlled by the variance of the random parameters: the smaller the variance, the better the approximation [32].

Additional discussions regarding the discretization of a random field may be found in the paper by Vanmarcke and Grigoriu [33] who evaluated the second-order statistics of the deflection of a beam whose rigidity varies randomly along its axis. In order to characterize the spatial correlation of the random material property, they defined a single parameter called the scale of fluctuation. The spatial discretization of the stochastic functions and the associated finite element mesh size depends on the statistical correlation of the underlying random field. The mesh should be fine enough for accuracy of computation but large with respect to the correlation length. The use of two meshes for computational efficiency, one depending on the structural topology and the other on the correlation length, was proposed by Der Kiureghian [34].

The development of a SFEM based on second-order perturbation expansion for static and transient problems was presented in a series of papers by Liu, Belytschko and Mani [35-38]. In Reference [35] the authors presented one of the first consistently derived SFEM for structural systems with uncertainties described by discrete random variables. The mean value and variance

12

of the response vector is obtained as the sum of the response evaluated at the mean value of the random parameters and second order correction terms. Both vectors were obtained by time integration of the equation of motion. The second order correction terms are defined in terms of sensitivity derivatives and each derivative is obtained integrating the equations of motion. The authors later extended their work to nonlinear structural dynamics problems with random fields [36]. The probabilistic characterization of the random field can be defined in terms of a covariance matrix if the random field is discretized with the shape functions used for the discretization of the displacement field. In Reference [37] the same authors presented a development of the SFEM based on the potential energy principle that permits to account for randomness in the geometric properties such as the shape of the domain and boundary conditions. They also presented a formulation for static analysis of continuous media with large deformations. Several aspects of the computational methodology were addressed [36-38]. For instance, they showed that a reduction in computational effort can be obtained transforming the non-diagonal covariance matrix of the discretized random variables into a diagonal variance matrix. This transformation comprises the solution of an eigenproblem similar to that employed in modal analysis in structural dynamic problems. Only a few modes are required to capture the major characteristics of the probabilistic distribution of the underlying random field. However, contrary to the modal analysis method of structural dynamics where only the lowest eigenvalues are used, the highest eigenvalues have to be employed.

Additional discussions regarding the application of SFEM for transient analysis can be found in the work by Liu, Besterfield and Belytschko [39,40]. In the application of this method for dynamic analysis, secular (unbounded) terms arise in the higher-order equations causing erroneous results. The authors presented a methodology to eliminate these secularities via Fourier analysis. In a later work, the same authors presented an application of the three-field Hu-Washizu Variational Principle as a basis to the SFEM [41]. The motivation for using this variational principle is that it permits to incorporate in a consistent manner into the analysis, probabilistic distributions for compatibility conditions, constitutive laws, the shape of domain and boundary conditions. The three stationary conditions of the resulting probabilistic variational principle are used to obtain the zeroth-, first-, and second-order variations in displacements, strains, and stresses; which can be later used to compute their statistics.

Nakagiri, Hisada and Toshimitsu [42] applied the SFEM to time-history analysis of a linear multi-degree-of-freedom system with uncertain damping. The equations of motion were decoupled making use of the orthogonality properties of the eigenvectors of the system. Since both the mass and stiffness matrices were considered to be deterministic, the eigenproperties also remained deterministic. The randomness in the damping values was introduced through modal damping ratios in the decoupled equations of motion. These equations were expanded following a second-order perturbation approach with the coefficients of the expansion obtained by time-integration. The calculation of mean and variance displacement time-histories for a deterministic ground acceleration excitation were selected to present numerical examples.

In a more recent work, Kareem and Sun [43] also considered the time response of structural systems with uncertain damping using a second-order perturbation technique and introducing uncertain damping ratios in the decoupled equations of motion. In Kareem and Sun's work

13

the excitation is assumed to be represented by a Gaussian white noise process and hence it is possible to obtain a closed-form solution for the problem. Both a time and a frequency domain formulation were presented. The numerical results demonstrate that the uncertainty in damping does influence the system response, although the effect is not very pronounced. Depending on the mean value of the damping ratio, the effects are more pronounced as the variance of the damping values increases.

The accuracy and range of application of SFEM may be verified by means of a Monte-Carlo numerical simulation (MCS). This is another very powerful and general approach for the solution of random equations. It is particularly useful for complex problems where numerous random variables are related through nonlinear relationships. Its applications are limited, however, by the expensive computational effort associated with its implementation. Compared to Monte-Carlo simulations, the computational requirements of the SFEM are often one order of magnitude smaller [37]. The first step in the Monte-Carlo solution of random equations is the generation of a set of random numbers for each random parameter in the equations. The deterministic equations of motion are solved by using each number in the set and the corresponding solutions are then analyzed statistically [44]. The means and variances of the response quantities tend to the exact solution as the size of the samples of random parameters becomes infinitely large. Examples of the application of the Monte-Carlo technique can be found in References [45] and [46]. The accuracy of SFEM, as compared with Monte-Carlo simulations, is well documented in several numerical applications involving trusses, bars, beams, and plates [35,36,37,38,41].

Shinozuka and Deodatis [47] proposed to use a Neumann expansion to study the response variability resulting from the spatial variation of the material properties of structures subjected to deterministic static loads. Instead of beginning with the solution expressed as a truncated series, the inverse of the operator in the governing equations (the stiffness matrix in this case) is approximately obtained in terms of a Neumann expansion. Replacing the approximate inverse operator in the equations of the system, the solution vector is at this stage expressed in a series form. An axially loaded rod with a Young's modulus which varies randomly in length was analyzed using the finite element method along with a first-order expansion of the inverse stiffness matrix of the system. In a follow-up paper, Yamazaki, Shinozuka and Dasgupta [48] used the Neumann expansion approach and the perturbation method to carry out a static finite element analysis of a plane-stress model considering a two-dimensional spatial variation of the material properties. The Neumann expansion was used in conjunction with the Monte-Carlo Simulation in a technique that they referred to as the Expansion MCS. The perturbation method was compared against the Expansion MCS and the Direct MCS (i.e., a simulation applied to the original random system). The numerical examples showed that the results obtained with the Expansion MCS compared very well with those obtained from the Direct MCS while requiring less computational time. The first and second order perturbation analysis also showed good agreement with the MCS when the range of variability of the properties is small.

To study the effect of the uncertainties in the structural parameters on the natural frequencies and modes of vibration of the system, one needs to solve an eigenvalue problem in which the elements of the system matrices are random variables. The solution of this problem, referred to as the Random Eigenvalue Problem, are eigenvalues and eigenvectors which also random vari-

14

ables. The solution of random eigenvalue problems is a relatively new and challenging problem in applied mathematics that has attracted the attention of both mathematicians and engineers [45,49,50,51,52,53]. There are several solution methods available, such as asymptotic methods [49], integral equation methods [54], hierarchy methods [55], and perturbation methods. A rigorous treatment of the subject is presented in the monograph by Scheidt and Purkert [51]. They only used the perturbation technique as the solution method because, as they pointed out, the other methods can only be used on a limited scale. The authors examined first the standard eigenvalue problem defined in terms of a symmetric deterministic matrix and a random perturbation matrix that satisfies the condition of being sufficiently "small" in a certain norm. It is possible to show that, under these conditions, the perturbation expansions for eigenvalues and eigenvectors exist and converge. Closed-form expressions for the coefficients of these expansions up to the fourth order are given. They computed approximate values for the expectations and correlations in terms of previously defined correlations between the coefficients of the perturbation matrix. The generalized eigenvalue problem, which is the form of interest in structural dynamics problems, is also examined using the same approach considered previously to determine the expansions for the eigenproperties. However, the correlation relations were not computed explicitly in this case. Mathematically rigorous discussions of several limit and existence theorems as well as a study of the eigenvalue problem defined in terms of random differential operators are also included in the monograph.

It should be pointed out most of the studies on random eigenproblems are limited to the calculations of the first moments of the eigenvalues and eigenvectors. The calculation of the probability distribution of the eigenvalues and eigenvectors is a problem that has been solved only for a very few simple cases [51,56]. It was shown [51], however, that for the case in which the random quantities involved in the eigenproblem are weakly correlated, the probability distribution function of the eigenproperties tends to be Gaussian.

One of the first studies on the solution of the eigenvalue problems with random matrices is due to Collins and Thomson [50]. They obtained expressions to define the differentials of the eigenvalues and eigenvector components. Based on these equations, they obtained the eigenvalue and eigenvector statistics as linearized expressions in terms of small variations about their mean values. The eigenvalue and eigenvectors associated with the longitudinal oscillations of a fixed-fixed rod and with a chain of equal springs and masses were calculated using the expressions developed. The validity of the formulation was confirmed by comparing the results with a Monte-Carlo simulation. In order to consider the correlation of the cross sectional area of the rod, they introduced a correlation coefficient matrix in which the elements are exponentials that decrease with the distance between the segments.

Shinozuka and Astill [45] obtained estimates of the variance of the n-th natural frequency of vibration of a beam-column with random properties using a Monte-Carlo simulation. They considered the special case of a continuous model of a simply supported beam-column with rotatory springs. The spring constants and the static axial load were considered independent random variables with zero mean whereas the material properties (Young's modulus and material density) and geometric properties (cross sectional area and moment of inertia) were zero mean correlated homogeneous random functions. The results obtained using the MCS were compared

15

with the corresponding results using the perturbation method. It was found that the perturbation method provides a reasonable solution over a much wider range of variation of the material and geometric properties than would be found in practice.

Hasselman and Hart [52] presented a method for computing the variance of the eigenproperties of large structural systems using a modal synthesis technique, which is compatible with Hurty's formulation of component mode synthesis [57]. The method is based on a first-order perturbation approach. The effects of modal truncation on the accuracy of the modal statistics were investigated and the results showed that component mode synthesis can be effectively used to compute the mean values and standard deviations of the eigenproperties. It was noted that the mean eigenvalues are the fastest to converge, followed by the mean eigenvectors, eigenvalue standard deviations and lastly by the standard deviations of the eigenvectors. The standard deviations of the eigenvalues and eigenvectors depend not only on the accuracy of the mean eigenvectors, but also on how well the derivatives of component eigenvectors are represented by the particular reduced set of component modes used in the synthesis.

Hart also examined in a following paper [53] the problem of calculating natural frequencies and vibration modes of a structure acted upon by external static loading. This results in the structure being stressed for the eigenvalue analysis. The complexity of the problem is compounded by the fact that in order to formulate the geometric stiffness matrix for the structure it is necessary to solve first a static problem to obtain the values of axial forces in structural members. Therefore, if the structural parameters are random variables, there are two levels of randomness involved in the problem. The first level is associated with the uncertainty in the values of the axial forces used to compute the geometric stiffness matrix. The second level is associated with the solution of the eigenvalue problem. The numerical examples showed that the results obtained with the first-order perturbation model compared well with those obtained via a Monte-Carlo simulation. It was observed that as the system loading approaches the buckling load, the uncertainty in the system fundamental natural frequency also increases. Therefore, a probabilistic eigenvalue analysis is advocated for the design of a structure with highly stressed members.

1.3   Scope and Organization

The objectives of the present study is to investigate the dynamic properties (natural frequencies and modal shapes) and the dynamic response of structural systems with random flexible connections. The study is divided into two parts. In the first part the problem is examined from a deterministic point of view. The second part examines the problem from a stochastic point of view. The report is also divided in two parts: the deterministic analysis is presented in Part I and the stochastic analysis in Part II. Each part contains three chapters.

In Chapter 1 of the first part a finite formulation is developed to incorporate the flexibility and the eccentricity of the connections. The flexible connections are modelled by rotational springs with linear moment-rotation relationships. The stiffness of the connections are defined in terms of non-dimensional "fixity factors". The effect of the connections on the mass and stiffness matrices of beam elements are represented by correction matrices that are added to the standard matrices. Numerical examples are presented to examine the influence of the connection characteristics on

16

the dynamic properties and the response to different dynamic loads. As an application example, the design response quantities of a building with flexible joints subjected to a seismic excitation defined in terms of ground response spectra are calculated.

Chapter 2 of Part I contains a Sensitivity Analysis of structures in which the stiffness of the connections are deterministic quantities. The objective is to develop closed form expressions to calculate the derivatives of eigenvalues and eigenvectors with respect to the fixity factors based on a second order perturbation expansions. These expressions will constitute the basis for the stochastic analysis undertaken in the second part. Part I of the report concludes with Appendix A which contains the final expressions of the eigenvalue and eigenvector derivatives.

In Part II of the report the stiffness of the flexible connections are regarded as random variables and a stochastic analysis of the problem is carried out. A Stochastic Finite Element Model (SFEM) is develop to take into account in a rational and systematic way the inherent uncertainty in the stiffness coefficients of the connections. In Chapter 1 the SFEM is used to characterize the probabilistic nature of the system eigenproperties. This requires the solution of a random eigenvalue problem. Second order perturbation expansions are used to calculate the statistics of the random eigenproperties. To validate the formulation and to assess its accuracy, numerical examples are presented in which the perturbation-based results are compared against Monte-Carlo simulations.

Chapter 2 of Part 2 address the problem of the calculation of the statistics of the time history response of structures with random flexible joints subjected to deterministic excitations. It is shown that the solutions based on straightforward expansions contain unbounded terms. A solution methodology based on the method of multiple scales that is free of the problems associated with the straightforward expansion is formulated. The formulation to calculate the expected value and variance of the response of multi degree of freedom systems subjected to a simple loading case is presented.

The final chapter of the report presents a summary and conclusions and some suggestions for future work.

Chapter 2

Formulation of the Finite Element Model

2.1 Introduction

In this chapter a finite element model of a beam element that incorporates the effects of the flexibility and eccentricity of the end connections on the dynamic behavior is introduced. Although different versions of the stiffness matrix of a beam with flexible end restraints have been obtained before, herea different approach is taken to define the finite element matrices in such a form that the effect of the connection size and flexibility can be clearly identified and separated. The formulation presented also provides the proper form of the consistent mass matrix for the first time.

2.2 Effect of the Connection Flexibility

For the dynamic finite element analysis of a two dimensional beam the displacement field $W(x,t)$ is usually expressed in terms of four shape functions $N_i(x)$ and the end displacements $w_1(t)$, $\theta_1(t)$, $w_2(t)$, $\theta_2(t)$ (see Figure 2.1):

$$W(x,t) = [N(x)] \left\{ \begin{array}{c} W_1(t) \\ \theta_1(t) \\ W_2(t) \\ \theta_{2(t)} \end{array} \right\} = [N(x)]\, u(t) \tag{2.1}$$

where:

$$[N(x)] = [N_1(x); N_2(x); N_3(x); N_4(x)] \tag{2.2}$$

with:

$$N_1(x) = 1 - 3\frac{x^2}{L^2} + 2\frac{x^3}{L^3} \tag{2.3}$$

$$N_2(x) = x - 2\frac{x^2}{L} + \frac{x^3}{L^2} \tag{2.4}$$

18

$$N_3(x) = 3\frac{x^2}{L^2} - 2\frac{x^3}{L^2} \tag{2.5}$$

$$N_4(x) = -\frac{x^2}{L} + \frac{x^3}{L^2} \tag{2.6}$$

One could, in principle, use this displacement field to develop a finite element model for beam member with flexible joints at both ends. However, since the end rotations of two members concurring at a node are different, such a formulation would increase the number of degrees of freedom of the structural model. It is therefore, more convenient to formulate a finite element model in which the rotational coordinates are the joint rigid body rotations ϕ_1 and ϕ_2 instead of the usual end member rotations θ_1 and θ_2 (see Figure 2.1). The rotations θ_i and ϕ_i are related as follows:

$$\phi_i = \theta_i + \alpha_i \; ; \; i = 1,2 \tag{2.7}$$

where α_i is the additional end rotation induced by the flexibility of the joint. The rotations α_i depend on the rotational stiffness K_i and bending moment M_i :

$$\alpha_i = -\frac{M_i}{K_i} \; ; \; i = 1,2 \tag{2.8}$$

Substituting equations (2.7) and (2.8) in equation (2.1), the displacement field can be written as:

$$W(x,t) = [N(x)]\{v(t) + p(t)\} \tag{2.9}$$

where:

$$v(t) = \left\{ \begin{array}{c} W_1(t) \\ \theta_1(t) \\ W_2(t) \\ \theta_2(t) \end{array} \right\} \quad ; \quad p(t) = \left\{ \begin{array}{c} 0 \\ -\frac{M_1(t)}{K_1} \\ 0 \\ -\frac{M_2(t)}{K_2} \end{array} \right\} \tag{2.10}$$

We need to express the end-of-member moments M_1 and M_2 in terms of the node displacements. This can be done substituting the assumed displacement field $W(x,t)$ in:

$$M_1(t) = E I w''(x,t)|_{x=0} \quad ; \quad M_2(t) = - E I w''(x,t)|_{x=L} \tag{2.11}$$

This leads to:

$$M_1 - E I \left[N''(x)\right]_{x=0} p = E I \left[N''(x)\right]_{x=0} v \tag{2.12}$$

$$M_2 + E I \left[N''(x)\right]_{x=L} p = - E I \left[N''(x)\right]_{x=L} v \tag{2.13}$$

The second derivative of the shape function matrix can be obtained from equations (2.3- 2.6):

19

$$\left[N''(x)\right]_{x=0} = \frac{1}{L^2} \left[-6 , -4L , 6 , -2L\right] \tag{2.14}$$

$$\left[N''(x)\right]_{x=L} = \frac{1}{L^2} \left[6 , 2L , -6 , 4L\right] \tag{2.15}$$

We define two nondimensional parameters γ_1 and γ_2 as follows:

$$\gamma_1 = \frac{EI}{L} \frac{1}{K_1} \quad : \quad \gamma_2 = \frac{EI}{L} \frac{1}{K_2} \tag{2.16}$$

These parameters characterize the linear behavior of the connection, and their values vary from zero for a perfectly rigid connection to infinity for a frictionless pin connection. Using equations (2.10) and (2.11), equations (2.12-2.13) can be written in matrix form as:

$$\begin{bmatrix} 1+4\gamma_1 & 2\gamma_2 \\ 2\gamma_1 & -1-4\gamma_2 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} =$$

$$= \frac{EI}{L^2} \begin{bmatrix} -6 & -4L & 6 & -2L \\ -6 & -2L & 6 & -4L \end{bmatrix} \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} \tag{2.17}$$

Solving for M_1 and M_2 we obtain:

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \frac{1}{D} \frac{EI}{L} [A] \begin{Bmatrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{Bmatrix} \tag{2.18}$$

where:

$$D = (1+4\gamma_1)(1+4\gamma_2) - 4\gamma_1\gamma_2 \tag{2.19}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \tag{2.20}$$

and the elements a_{ij} are:

$$a_{11} = -a_{13} = -\frac{6}{L}(1+2\gamma_2) \tag{2.21}$$

$$a_{12} = -4(1+3\gamma_2) \tag{2.22}$$

$$a_{14} = -2 \tag{2.23}$$

20

$$a_{21} = -a_{23} = -\frac{6}{L}(1 + 2\gamma_1) \qquad (2.24)$$

$$a_{22} = -2 \qquad (2.25)$$

$$a_{24} = -4(1 + 3\gamma_1) \qquad (2.26)$$

Equation (2.18) permits to express the vector p defined in equation (2.10) in terms of the vector of end displacements v, as follows:

$$p = \frac{1}{D}[S]\,v \qquad (2.27)$$

where:

$$[S] = \begin{bmatrix} \mathbf{0}^T \\ S_1^T \\ \mathbf{0}^T \\ S_2^T \end{bmatrix} \qquad (2.28)$$

and:

$$S_1^T = \gamma_1\,[a_{11},a_{12},a_{13},a_{14}] \qquad (2.29)$$

$$S_2^T = \gamma_2\,[a_{21},a_{22},a_{23},a_{24}] \qquad (2.30)$$

Substituting equation (2.27) in equation (2.9) we obtain the displacement field $W(x,t)$ in terms of the end displacements W_1, W_2 and the total end rotations φ_1, φ_2:

$$W(x,t) = [N(x)]\left[[I] + \frac{1}{D}[S]\right]v(t) \qquad (2.31)$$

The presence of the matrix $[S]$ in the above equation reflects the effect of the connection flexibility. If the connections are rigid, then all elements of $[S]$ are zero and equation (2.31) reduces to the classical definition of the displacement field in terms of the nodal displacements. The use of equation (2.31) in the development to be presented in the sequel will enable us to identify distinctly the contribution of the connection flexibility to the finite element matrices.

2.3 Effect of the Connection Eccentricity

In some cases the size of the end connections may not be negligible and its effects should be incorporated in the finite element model. The connections are represented in Figure 2.1 as short rigid bars of length l_1 and l_2. When l_1 and l_2 are not small values, the displacements w_1, W_1 and w_2, W_2 are approximately related as follows:

21

$$w_1 = W_1 + \phi_1 l_1 \tag{2.32}$$

$$w_2 = W_2 + \phi_2 l_2 \tag{2.33}$$

Introducing a new displacement vector q:

$$q(t) = \left\{ \begin{array}{c} W_1(t) \\ \phi_1(t) \\ W_2(t) \\ \phi_2(t) \end{array} \right\} \tag{2.34}$$

and considering equations (2.32- 2.33), we can relate the vectors $q(t)$ and $v(t)$ as follows:

$$v(t) = [[I] + [X]]\, q(t) \tag{2.35}$$

where:

$$[X] = \begin{bmatrix} 0 & l_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -l_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.36}$$

The separation of the transformation matrix relating $v(t)$ and $q(t)$ in equation (2.35) into an identity matrix $[I]$ and another matrix $[X]$ is done to keep the contribution of the connection size to the finite element matrices separately identified.

Substituting equation (2.35) in equation (2.31) one obtains the displacement field in terms of the new end displacement vector:

$$W x, t) = [N(x)]\, [[I] + [R]]\, q(t) \tag{2.37}$$

where the matrix $[R]$ is defined by:

$$[R] = \frac{1}{D}\, [[S] + D[X] + [S][X]] \tag{2.38}$$

This matrix takes into account the combined effect of the flexibilities and eccentricities of the connections at the two ends. The first matrix in the definition of $[R]$ reflects the effect of connection flexibility only, the second matrix reflects the effect of connection eccentricity only, and the last matrix introduces the combined effect of the flexibilities and eccentricities of the end connections. By setting one or more of these matrices to zero, the corresponding effects can be conveniently eliminated from the response computations. Substituting for $[S]$ and $[X]$ from equations (2.28) and (2.36), respectively, into equation (2.38), and after carrying-out the matrix products, we obtain $[R]$ in explicit form as:

22

$$[R] = \frac{1}{D} \begin{bmatrix} 0 & r_{12} & 0 & 0 \\ r_{21} & r_{22} & r_{23} & r_{24} \\ 0 & 0 & 0 & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix} \qquad (2.39)$$

The elements r_{ij} are:

$$r_{12} = f_1 \, D \, L \qquad (2.40)$$

$$r_{21} = -r_{23} = \gamma_1 \, a_{11} \qquad (2.41)$$

$$r_{22} = \gamma_1 \, a_{12} + 6 \, f_1 \, \gamma_1 \, (1 + 2 \, \gamma_2) \qquad (2.42)$$

$$r_{24} = \gamma_1 \, a_{14} + 6 \, f_2 \, \gamma_1 \, (1 + 2 \, \gamma_2) \qquad (2.43)$$

$$r_{34} = -f_2 \, D \, L \qquad (2.44)$$

$$r_{41} = -r_{43} = \gamma_2 \, a_{21} \qquad (2.45)$$

$$r_{42} = \gamma_2 \, a_{22} - 6 \, f_1 \, \gamma_2 \, (1 + 2 \, \gamma_1) \qquad (2.46)$$

$$r_{44} = \gamma_2 \, a_{44} + 6 \, f_2 \, \gamma_2 \, (1 + 2 \, \gamma_1) \qquad (2.47)$$

The two coefficients f_1 and f_2 are non dimensional parameters characterizing the joint size and defined as follows:

$$f_1 = \frac{l_1}{L} \quad ; \quad f_2 = \frac{l_2}{L} \qquad (2.48)$$

These parameters will be referred to as the eccentricity ratios of the element.

2.4 Equations of Motion

The equations of motion for a vibrating beam element with flexible connections of finite length will be obtained using the Lagrange's equations for a discrete parameter system:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial U}{\partial q} = Q(t) \qquad (2.49)$$

in which:

q: Vector of generalized coordinates. In this case, the set of generalized coordinates are the joint displacements in equation (2.34).
T: Total kinetic energy of the element.
U: Total elastic potential energy of the element.
Q: Vector of generalized non conservative forces.

To apply equation (2.49) we need to express the energy terms of the beam element as function of the generalized coordinates. The total kinetic energy of the element takes the form:

$$T = \frac{1}{2} \int_0^L m(x) \, \dot{W}(x,t)^2 \, dx \qquad (2.50)$$

23

and considering equationt(2.37) it is possible to write:

$$T = \frac{1}{2} \dot{q}(t)^T [[M_0] + [M_1]] \dot{q}(t) \tag{2.51}$$

where the matrices $[M_0]$ and $[M_1]$ are defined as:

$$[M_0] = \left[\int_0^L m(x) [N(x)]^T [N(x)] \, dx \right] \tag{2.52}$$

$$[M_1] = [R]^T [M_0] [R] + [M_0] [R] + [[M_0] [R]]^T \tag{2.53}$$

The matrix $[M_0]$ is the conventional "consistent" mass matrix for a beam element, and the matrix $[M_1]$ is a correction matrix due to both the flexibility and finite size of the connection. Using the definition *of* $[R]$ in equation (2.38), it is possible to identify the contributions of the connection flexibilities and eccentricities to this correction matrix. It is noted that the fact that a new mass matrix is required for the dynamic analysis of a structure with flexible connections was missed in the previous studies. Moreover, the change in the mass matrix due to the combined effects of the flexibility and the eccentricity of the connection has not been reported before. It should be realized that the modification of the mass matrix is only due to the set of end coordinates used, since it is obvious that the massless rotational springs in the beam model cannot affect the total kinetic energy.

Now we will turn our attention to the potential energy U in order to obtain the stiffness matrix of the finite element. The potential energy of a beam with flexible connections is composed of two terms:

$$U = U_e + U_s \tag{2.54}$$

The first term U_e accounts for the elastic deformation of the element:

$$U_e = \frac{1}{2} \int_0^L E I \left(\frac{\partial^2 W(x,t)}{\partial x^2} \right) dx \tag{2.55}$$

The second term U_s takes into account the rotational deformation of the springs modeling the flexible connections:

$$U_s = \frac{1}{2} K_1 \alpha_1^2 + \frac{1}{2} K_2 \alpha_2^2 \tag{2.56}$$

Substituting the displacement field $W(x,t)$ from equation (2.37) in equation (2.55) yields:

$$U_e = \frac{1}{2} q(t)^T [[K_0] + [K_1]] q(t) \tag{2.57}$$

where:

$$[K_0] = \left[\int_0^L E I \left[N''(x) \right]^T \left[N''(x) \right] dx \right] \tag{2.58}$$

24

$$[K_1] = [R]^T [K_0] [R] + [K_0] [R] + [[K_0] [R]]^T \tag{2.59}$$

According to the above definitions, $[K_0]$ represents the standard stiffness matrix of a beam element, whereas $[K_1]$ is a correction accounting for the presence of flexible finite-size connections at the ends of the element.

Now we need to express the potential energy of the rotational springs U_s in terms of the generalized coordinates. Using equations(2.8), (2.10), (2.27) and (2.28) we can write:

$$\alpha_1 = -\frac{M_1}{K_1} = -\frac{1}{D} S_1^T \boldsymbol{v} \tag{2.60}$$

$$\alpha_2 = -\frac{M_2}{K_2} = -\frac{1}{D} S_2^T \boldsymbol{v} \tag{2.61}$$

And using equation (2.35) to take into account the connection finite size we obtain:

$$\alpha_1 = -\frac{1}{D} R_2^T \boldsymbol{q} \tag{2.62}$$

$$\alpha_1 = -\frac{1}{D} R_4^T \boldsymbol{q} \tag{2.63}$$

where $\boldsymbol{R}_2$ is a vector with elements $r_{21}, r_{22}, r_{23}, r_{24}$ defined in equations (2.41 - 2.43), and $\boldsymbol{R}_4$ is a vector with elements $r_{41}, r_{42}, r_{43}, r_{44}$ defined in equations (2.45 - 2.47).

Substituting for α_1, α_2 from equation equations (2.60 - 2.61) in equation (2.55), the potential energy U_s can be expressed as:

$$U_s = \frac{1}{2} \boldsymbol{q}^T \left[\frac{K_1}{D^2} R_2 R_2^T + \frac{K_2}{D^2} R_4 R_4^T \right] \boldsymbol{q} \tag{2.64}$$

Finally, the total potential energy U in equation (2.54) is obtained adding the terms in equations (2.57) and (2.64):

$$U = \frac{1}{2} \boldsymbol{q}(t)^T [[K_0] + [K_1] + [K_2]] \boldsymbol{q}(t) \tag{2.65}$$

where:

$$[K_2] = \left[\frac{K_1}{D^2} R_2 R_2^T + \frac{K_2}{D^2} R_4 R_4^T \right] \tag{2.66}$$

Substituting for T and U from equations (2.51) and (2.65) in the Lagrange's equation (2.49) and carrying out the derivatives, the equations of motion for a beam with flexible end connections become:

$$[[M_0] + [M_1]] \ddot{\boldsymbol{q}}(t) + [[K_0] + [K_1] + [K_2]] \boldsymbol{q}(t) = \boldsymbol{Q}(t) \tag{2.67}$$

where $\boldsymbol{Q}(t)$ is the vector of generalized non conservative forces associated with the nodal coordinates $\boldsymbol{q}$, we need to calculate the virtual work δW_{nc} of the distributed force field $f(x,t)$ acting on the beam through a virtual displacement $\delta W(x,t)$. From equation (2.37) we can write:

25

$$\delta W_{nc} = \int_0^L f(x,t) \; \delta W(x,t) \; dx$$

$$= \left[\int_0^L f(x,t) \; [N(x)] \; [[I] + [R]] \; dx \right] \delta q \tag{2.68}$$

The elements of the vector of generalized forces are the terms multiplying each generalized coordinate q_i in the expression of the virtual work. Therefore, writing equation (2.68) as:

$$\delta W_{nc} = Q(t)^T \delta q \tag{2.69}$$

and comparing equations (2.68) and (2.69), we conclude that the vector of generalized forces $Q(t)$ is obtained by adding the contributions of two terms:

$$Q(t) = Q_0(t) + Q_1(t) \tag{2.70}$$

where the first term is the standard force vector for a beam element:

$$Q_0(t) = \int_0^L f(x,t) \; [N(x)]^T \; dx \tag{2.71}$$

The second term in equation (2.70) stems from the presence of the flexible connections and it is defined as:

$$Q_1(t) = \int_0^L f(x,t) \; dx \; [R]^T \; [N(x)]^T \, dx \tag{2.72}$$

2.5 Explicit Form of the Element Matrices

The element matrices will be only defined for the flexural effects (transverse displacements and rotations) since the connections are assumed to be rigid for the axial and torsional displacements. The matrices for a three-dimensional beam element can be obtained by combining the matrices associated with the flexural effects in the two principal planes of the cross section and adding the standard matrices for axial and torsional displacements.

To obtain more compact expressions for the correction matrices, we will define a new set of nondimensional parameters to characterize the connection flexibility. These parameters,referred to as the fixity factors of the element, are:

$$\mu_1 = \frac{1}{1 + 3\gamma_1} \qquad ; \qquad \mu_2 = \frac{1}{1 + 3\gamma_2} \tag{2.73}$$

The subscripts 1 and 2 designate the two nodes of the element, and γ_1, γ_2 are the nondimensional parameters defined in equation (2.16). For the three-dimensional case, it is necessary to define pairs of fixity factors at both ends of the element, corresponding to each principal plane of the cross-section.

2.5.1 Mass Matrix

The mass matrix for a element with flexible connections of finite size is defined in equation (2.67) as the sum of two parts. The first component is the consistent mass matrix, which for an element with constant cross sectional area and mass density can be written as follows:

$$[M_0] = \frac{m\,L}{420} \begin{bmatrix} 156 & & & symmetric \\ 22\,L & 4\,L^2 & & \\ 54 & 13\,L & 156 & \\ -13\,L & -3\,L^2 & -22\,L & 4\,L^2 \end{bmatrix}$$

(2.74)

in which:

m: Distributed mass per unit of length.
L: Length of the element.

To obtain the explicit form of the correction matrix is necessary to carry out the matrix products indicated in equation (2.53). The algebra involved makes this task very burdensome. Therefore, a computer program with symbolic algebra capabilities [58] was employed for this purpose. The final expression of $[M_1]$ after all the matrix manipulations are carried out is:

$$[M_1] = \frac{m\,L}{420} \begin{bmatrix} -156\,\beta_1\,(\mu_1,\mu_2) & & \\ -22\,L\,\beta_3\,(\mu_1,\mu_2,f_1) & -4\,L^2\,\beta_5\,(\mu_1,\mu_2,f_1) & \\ 54\,\beta_2\,(\mu_1,\mu_2) & -13\,L^2\,\beta_4\,(\mu_1,\mu_2,f_2) & \\ 13\,L\,\beta_4\,(\mu_2,\mu_1,f_2) & 3\,L^2\,\beta_6\,(\mu_1,\mu_2,f_1,f_2) & \end{bmatrix}$$

$$\begin{bmatrix} & & symmetric \\ -156\,\beta_1\,(\mu_2,\mu_1) & \\ 22\,L\,\beta_3\,(\mu_2,\mu_1,f_2) & -4\,L^2\,\beta_5\,(\mu_2,\mu_1,f_2) \end{bmatrix}$$

(2.75)

where the following auxiliary functions were introduced:

$$\beta_1(a,b) = \Big(64 + 7\,a^2b^2 + 55\,a^2b - 50\,ab^2 - 32\,(a^2 + b^2) +$$
$$+ \; 16\,ab - 224\,a + 196\,b\Big) / \Big(39\,R(a,b)^2\Big)$$

(2.76)

$$\beta_2(a,b) = \Big(128 + 14\,a^2b^2 + 5\,(a^2b + ab^2) - 64\,(a^2 + b^2) +$$

27

$$+ \quad 32\,ab - 28\,(a\,+\,b)) \, / \left(39\,R(a,b)^2 \right) \tag{2.77}$$

$$\mathcal{J}_3(a,b,c) = 2 \left(88 - c\,\varepsilon_1\,(a,b) - 7\,a^2b^2 + 43\,a^2b - 16\,ab^2 - \right.$$
$$\left. - \quad 32\,a^2 + 36\,ab - 112\,a \right) / \left(11\,R(a,b)^2 \right) \tag{2.78}$$

$$\mathcal{J}_4(a,b,c) = 2 \left(104 - c\,\varepsilon_2\,(a,b) - 21\,a^2b^2 + 32\,a^2b + 19\,ab^2 - \right.$$
$$\left. - \quad 2\,ab + 64\,b^2 - 196\,b \right) / \left(13\,R(a,b)^2 \right) \tag{2.79}$$

$$\mathcal{J}_5(a,b,c) = 2 \left(16 - c^2\,\varepsilon_1\,(a,b) - c\,\varepsilon_3\,(a,b) - 7\,a^2b^2 + 31\,a^2b - \right.$$
$$\left. - \quad 32\,a^2 - 8\,ab \right) / \left(R\,(a,b)^2 \right) \tag{2.80}$$

$$\mathcal{J}_6(a,b,c,d) = (48 - 2\,c\,d\,\varepsilon_2\,(a,b) - c\,\varepsilon_4\,(a,b) - d\,\varepsilon_4\,(b,a) - $$
$$- \quad 28\,a^2b^2 + 64\,(a^2b\,+\,ab^2) - 148\,ab) / \left(3\,R(a,b)^2 \right) \tag{2.81}$$

in which the functions ε_1, ε_2, ε_3 and ε_4 are defined as:

$$\varepsilon_1(x,y) = 32\,x^2y^2 - 55\,x^2y + 50\,xy^2 + 32\,(x^2\,+\,y^2) - $$
$$- \quad 328\,xy + 224\,x - 196\,y + 560 \tag{2.82}$$

$$\varepsilon_2(x,y) = 41\,x^2y^2 + 5\,(x^2y\,+\,xy^2) - 64\,(x^2\,+\,y^2) - $$
$$- \quad 184\,xy - 28\,(x\,+\,y) + 560 \tag{2.83}$$

$$\varepsilon_3(x,y) = 25\,x^2y^2 - 86\,x^2y + 64\,x^2 + 32\,xy^2 - 160\,xy + $$
$$+ \quad 224\,x \tag{2.84}$$

$$\varepsilon_4(x,y) = 55\,x^2y^2 - 64\,x^2y - 38\,xy^2 - 100\,xy - 128\,y^2 + 392\,y \tag{2.85}$$

and the function R(a,b) is given by:

$$R(a,b) = 4 - ab \tag{2.86}$$

Figures (2.2.a-f) show the functions $\mathcal{J}_1$ to $\mathcal{J}_6$, for the case $c = d = 0$, that is, when both eccentricity ratios are zero. The functions take a zero value when the fixity factors are equal to one, that corresponds to the perfectly rigid joints case. The surfaces representing the functions $\mathcal{J}_2$ and $\mathcal{J}_6$ in Figures (2.2.c) and (2.2.f) are symmetric with respect to the vertical plane defined by $a - b = 0$. The two functions $\mathcal{J}_2$ and $\mathcal{J}_6$ are associated with the coefficients of the mass matrix that relate, respectively, the translational and rotational degrees of freedom at both ends of the element. All the functions have a unit value when the fixity factors represent a pin connection (that is, when $a = b = 0$), with the exception of the functions β_1 and $\mathcal{J}_2$. These two functions correspond to the coefficients of the mass matrix that relate translational coordinates.

2.5.2 Stiffness Matrix

The stiffness matrix for a element with flexible connections of finite size is given in equation (2.67) as the sum of three parts. The first component is the standard stiffness matrix which, for an element with uniform cross sectional area and homogeneous material, is defined as follows:

28

$$[K_0] = \frac{2EI}{L^3} \begin{bmatrix} 6 & & & \\ 3L & 2L^2 & & \\ -6 & -3L & 6 & \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix} \qquad (2.87)$$

Here again to obtain the explicit form of the stiffness correction matrix a symbolic algebra software [58] was used. The outcome of carrying-out the matrix products in equations (2.59) and (2.66) and combining the two resulting expressions is:

$$[K_1 + K_2] = \frac{2EI}{L^3} \begin{bmatrix} -6\,\alpha_1\,(\mu_1,\mu_2) & \\ -3\,L\,\alpha_2\,(\mu_1,\mu_2,f_1) & -2\,L^2\,\alpha_3\,(\mu_1,\mu_2,f_1) \\ 6\,\alpha_1\,(\mu_1,\mu_2) & 3\,L\,\alpha_2\,(\mu_1,\mu_2,f_1) \\ -3\,L\,\alpha_2\,(\mu_1,\mu_2,f_2) & -L^2\,\alpha_4\,(\mu_1,\mu_2,f_1,f_2) \end{bmatrix}$$

$$\begin{matrix} symmetric & \\ -6\,\alpha_1\,(\mu_1,\mu_2) & \\ 3\,L\,\alpha_2\,(\mu_2,\mu_1,f_2) & -2\,L^2\,\alpha_3\,(\mu_2,\mu_1,f_2) \end{matrix} \qquad (2.88)$$

where the auxiliary functions α_i are defined as follows:

$$\alpha_1(a,b) = (4 - 2ab - a - b) \,/\, R(a,b) \qquad (2.89)$$

$$\alpha_2(a,b,c) = 2\,(2 - abc - ac - bc - ab - a) \,/\, R(a,b) \qquad (2.90)$$

$$\alpha_3(a,b,c) = \big(4 - 3c^2\,(ab + a + b) - 3abc - 6ac - $$
$$-\; ab - 3a\big) \,/\, R(a,b) \qquad (2.91)$$

$$\alpha_4(a,b,c,d) = (4 - 6cd\,(ab + a + b) - 3ab\,(c + d) - $$
$$-\; 6bc - 6ad - 4ab) \,/\, R(a,b) \qquad (2.92)$$

Figures (2.3.a-d) show the functions α_1 to α_4 , for the case $c = d = 0$. The four functions are equal to zero for the case of perfectly rigid joints. The functions α_1 and α_4, which are associated to coefficients relating the translational and rotational degrees of freedom at different ends of the element, present a symmetry with respect to the vertical plane defined by $a - b = 0$. For the pin connection case, the functions are equal to one, and the coefficients of the matrix defined in equation (2.88) become the opposite of the corresponding ones in equation (2.87). Therefore, for this case, the stiffness matrix of the element is the null matrix.

29

Note that since all the coefficients α_i that define the correction stiffness matrix are positive, the potential energy of a structural system with flexible joints is always less than for a system with rigid joints. Considering the definition of the eigenvalues in terms of the Rayleigh's quotient, one could conclude at first sight that the eigenvalues of the first system will always be less than those with rigid joints. However, the mass matrix and consequently the denominator of the Rayleigh's quotient is also modified by the presence of flexible connections. Moreover, some of the coefficients β_i that define the correction mass matrix have variable sign. Therefore, it is possible that some of the eigenvalues of the system with flexible joints, especially the higher ones, could be larger than those of the rigid joint system.

2.6 Numerical Examples

2.6.1 Example No. 1: One-Story Frame

The simple plane frame shown in Figure (2.4) will be used for the first numerical example. In order to obtain an accurate description of the dynamic behavior of the structure, each column was discretized with 12 elements and the beam was divided into 24 elements. The total number of degrees of freedom was 141. The flexible connections are those in the union of the beam with the columns, identified as joint (a) and joint (b) in the figure.

2.6.1.1 Effect of Flexible Connections on the Dynamic Properties

The variation in the values of the six lower frequencies with respect to the fixity factors is shown in Figure (2.5). The fixity factors are assumed to be identical for both connections. It is observed that all the frequencies diminish monotonically as the stiffness of the connections are reduced. It is interesting to note that curves corresponding to the 5th and 6th frequencies intersect at a fixity factor equal to 0.973. From this point on, the difference between the frequencies increases as the fixity factors decrease. In Figure (2.6) the curves of variation of the natural frequencies are plotted normalized by dividing them by the corresponding frequency of the structure with rigid joints. It can be seen that the 2nd frequency shows the maximum departure from the rigid joints reference. The reason for this seems to be that the second mode is associated to the deflection of the beam as a single element, as it can be seen in Figure (2.7.b). The relative variation seems to be less important for the higher frequencies, with the exception of the 5th frequency. The curve of this frequency is a nonlinear function of the fixity factor, as opposed to those of the remaining frequencies in which the variations are almost linear.

The modal shapes of the portal frame corresponding to the lower six natural frequencies are presented in Figures (2.7.a-f) for three different values of the fixity factor. The height and length of the structure were scaled by a factor such that the modal shapes can be plotted in the same figure. It can be seen that the higher the modes, the more pronounced the effect of the flexible connections in the modal shapes. In general, the reduction in the value of the connection stiffness has the effect of "smoothing" the modal shapes in different parts of the structure. For example, this can be observed in the beam for the 1st and 4th modes and in the columns for the 2nd

30

and 3rd modes. The flexural modal deformation in these elements diminish as the fixity factors decrease. The crossing of the 5th and 6th frequencies previously pointed out can be noticed in the corresponding modal shapes in Figures (2.7.e) and (2.7.f). For the structure with rigid joints the 5th mode is symmetric with the modal deflection pattern of the beam associated with flexural deformations. The 6th mode, on the other hand, is antisymmetric with the flexural deformation pattern associated to the columns. For the structure with flexible joints and with fixity factors lower than the value at the crossing point, for instance equal to 0.1 and 0.5, the modal shapes are interchanged with respect to the rigid case. This can be seen by comparing Figures (2.7.e) and (2.7.f).

2.6.1.2 Effect of Flexible Connections on the Dynamic Response

We will consider now the influence of the flexibility of the beam connections on the dynamic response of the structure. In all the numerical examples the modal damping ratio was assumed to be equal to 5% for all the modes, and both the initial displacements and velocities are assumed to be zero.

First the response to a a step loading function acting on joint (a) in the horizontal direction will be examined. Figure (2.8) shows the response time histories for the horizontal displacement of joint (a) for three values of the fixity factor. The displacements are normalized by dividing them by the maximum value of the displacement of the rigid joints case. A normalized time was also used to plot the response: the time ordinates were divided by the fundamental natural period of the rigid joints frame. The different response curves oscillate about the static equilibrium position with peaks that increase for the low fixity factors. The effect of the flexible connections in the natural frequencies can also be seen in this figure. According to Figure (2.6), when the values of μ are reduced from 1.0 to 0.10 the fundamental frequency of the frame decreases up to a 72% of the fixed case value. This change can be noticed in the curves of Figure (2.8) since due to the type of excitation it is the first mode that governs the response.

To emphasize the increment of the response due to the joint flexibility, Figure (2.9) displays the behavior in the initial instants of time. It can be observed that the maximum displacement is increased by 100% when the fixity factors are reduced from 1.0 to 0.1. This increment is even larger when the rotation of the joint (a) is considered. The time history of the response of joint (a) is plotted in Figure (2.10). The first part of the time history shown in Figure (2.11) reveals that the maximum rotation of the structure with fixity factors equal to 0.10 is more than three times that of the structure with rigid joints.

Secondly we consider the response of the structure to an impulsive load acting on joint (a) in the horizontal direction. Figures (2.12) and (2.14) show, respectively, the normalized time histories for the horizontal displacement and the rotation of joint (a) for three different values of the fixity factor. The effect of the flexibility of the beam connections in the natural frequencies and the peak values of the response can be seen in these figures. Figures (2.13) and (2.15) show the initial behavior of the time histories, where it can be observed that higher frequencies participate in the initial stage of the response.

31

2.6.2 Example No. 2: Two-story frame

In the next example we will examine the effect of the flexible connections on a non-symmetric plane frame. The structure, shown in Figure (2.16), was modelled with seventeen frame elements and a total of 42 dof. Only the connections between the beams and columns are assumed to be flexible. These are identified in the figure as joints (a) through (f).

2.6.2.1 Effect of Flexible Connections on the Dynamic Properties

Figure (2.17) shows the variation of the first three natural frequencies of the structure as a function of the values of the fixity factors of those joints. The fixity factors for joints (a) and (c) are assumed to be equal. The remaining connections are assumed to be all rigid in one case and all flexible with the same fixity factor equal to 0.7 in the other case. The solid lines represent the variation in the natural frequencies when all the joints, except (a) and (b), are rigid. The dotted lines correspond to a fixity factor equal to 0.7. The three frequencies decrease when the stiffness connections decrease. When the fixity factors of the other joints of the structure are reduced from 1.0 to 0.7, the curves have the same form but lower values. The variation in the frequencies normalized by dividing them by the corresponding rigid joint frequencies is shown in Figure (2.18).

Figure (2.19) shows again the variation of the lower three natural frequencies but this time the fixity factors of the connections (a),(b),(c) and (d) of the beams at the lower level are simultaneously varied. The remaining connections (e) and (f) are either assumed to be flexible with $\mu=0.7$ or perfectly rigid. Figure (2.20) displays the variation of the natural frequencies normalized as explained before. The figure indicates that the 2nd frequency is practically insensitive to the reduction in the stiffness of the connections, except for values of μ less than 0.35.

2.6.2.2 Effect of the Flexible Connections on the Response to a Harmonic Load

The two story frame is subjected to a harmonic load acting in the vertical direction at the middle of the beam between joints (a) and (b). Any steady state response quantity associated with a particular dof of a structural system subjected to a harmonically varying load at a given point can be completely defined by the (Complex) Frequency Response Function. For a linear system with N dof, the Frequency Response Function for the r-th dof due to a force applied at the s-th dof is defined as:

$$H_{rs}(\omega) = \sum_{j=1}^{N} \frac{\phi_{rj}\,\phi_{sj}}{\omega_j^2} \frac{1}{1 - \left(\frac{\Omega}{\omega_j}\right)^2 + 2i\,\xi_j\left(\frac{\Omega}{\omega_j}\right)} \tag{2.93}$$

where Ω is the excitation frequency.

Figure (2.21) shows the magnitude of the Frequency Response Function (FRF) corresponding to the vertical displacement of the point of application of the load. The joints (a),(b),(c) and (d) were assumed to be flexible and two values of the fixity factors were considered: 0.1 and 0.5. For comparison, the FRF for the frame with rigid joints is also shown in the figure. The driving frequency Ω in the horizontal axis was normalized by dividing it by the fundamental frequency

32

of the structure with rigid joints. The three curves were also normalized with respect to the peak value of the FRF for the rigid joints case. As it was expected, the maximum amplification occurs for the lower values of the fixity factor. Moreover, it can be observed that the frequency corresponding to the peak values of the FRF also decreases with the fixity factors. When $\mu = 0.1$. the maximum amplification occurs at the 6th frequency whereas for $\mu = 0.5$ and $\mu = 1.0$ this happens at the 7th frequency.

2.6.3 Example No. 3: Ten-Story Plane Frame

One of the transverse planar frameworks from a ten story unbraced steel building will be used to illustrate the effect of the joint characteristics on the dynamic properties and seismic response of practical structures. The structure is shown in Figure (2.22), along with its geometry and member properties. The model has 33 nodes and 90 dof. In addition to the distributed mass of the members, lumped masses of 2400 slugs due to the floor mass were added to each node. Only the beam-to-column connections were considered to be flexible and all the connections were assumed to be identical.

2.6.3.1 Effect of the Flexible Connections on the Dynamic Properties

Figure (2.23) shows the variation of the first five natural frequencies as a function of the fixity factors defined in equation (2.73). The natural frequencies were normalized by dividing their values by the frequencies corresponding to the structure with rigid connections. Observing the figure one can see that the flexibility of the connections has a more pronounced effect on the lower frequencies. This is an important characteristic for seismic analysis since the response of these structures is generally dominated by the lower modes. Moreover, since the response is also strongly influenced by the frequency content of the earthquake motion, a proper consideration of the flexibility of the joints is necessary to obtain the actual values of the natural frequencies because resonant phenomena may be otherwise unexposed.

The change in the first five participation factors with the joint flexibility is displayed in Figure (2.24), where the participation factors have been normalized with respect to the corresponding values for a structure with rigid connections. It is observed that the lower participation factor, and consequently the relative importance of the first mode on the total response, decreases as the flexibility of the connections increases. On the other hand, the participation factor of the higher modes increases with the joint flexibility. Hence, the relative contributions of the higher modes to the seismic response become more important as the beam-to-column connections become more flexible.

Figures (2.25.a-c) show how the different fixity factors alter the first three modal shape vectors. The modal shape vectors plotted in these figures are the horizontal displacements of the nodes located along the right column of the structure. Figure (2.25.a) shows that the effect of the joint flexibility on the first modal shape vector depends on the floor level considered. As expected. the modal displacement at the highest level increases when the flexibility of the connections decreases. However, this trend reverses for the lower floor levels. The joint flexibilities also affects the higher modal shapes in different way depending on the floor level considered. Figures

33

(2.25.b) and (2.25.c) show the second and third modal shape vectors calculated with three fixity factors. As it can be seen from these figures, the location of the nodal points also changes with the fixity factors.

In all the previous cases the size of the connections was considered to be negligible. The effect of finite size connections is studied next. Figure (2.26) shows how the fundamental frequency of the building is affected by the eccentricity ratio, defined in equation (2.48) as the ratio between the length of the connection and the member span. The three curves, which were obtained for fixity factor values of 0.1, 0.5 and 1.0, show that the first natural frequency increases with the eccentricity ratio. The rate of change in this frequency due to the finite connection effect is almost independent of the fixity factor, except for low values of this parameter. Inspection of Figure (2.26) reveals that the change in the natural frequencies is small: even for an eccentricity ratio of 0.1, the fundamental frequency increases only by 15% for a structure with fixity factors between 0.5 and 1. Figure (2.27) presents the variation of the second natural frequency as a function of the eccentricity ratios. In this case, the influence is even smaller than in the previous case.

It was shown before that in the finite element modeling of a framed structure with flexible joints the mass matrix must also be corrected to account for both the flexibility and finite-size characteristics of the connections. As pointed out earlier, this fact was ignored in previous studies. In building structures the effect of ignoring the correction mass matrix is not important because the lumped floor masses are the dominant terms in the global mass matrix of the structure. However, the effect could be important for other framed structures. To examine this effect we will consider only the skeleton of the same structure in Figure (2.22) without the lumped floor masses. The global mass matrix is then only made up of the conventional consistent mass matrices representing the distributed mass of the members. The lower 20 natural frequencies calculated with the model using the corrected and the conventional mass matrices are shown in Figures (2.28.a-d) considering four different fixity factors. Figure (2.29) summarizes the information and it shows the relative error in the calculated frequencies in each case, considering that the "exact" values are those obtained with corrected mass matrix. It is seen that this error varies with the frequency number and, at the same time, is larger for the lower fixity factors. For the first ten frequencies the difference is small, but for the higher frequencies the error increases, although not in a monotonic way. The reason for this behavior seems to be the following. The effects that the changes in the flexibility of the joints have on the mass matrix of the element can be evaluated examining the variation of each one of its terms. For the case of equal variations of both fixity factors, the coefficients more strongly affected, are those that relate rotational dofs with themselves and those that relate rotational with translational dofs. The effect in the terms of the mass matrix associated with the translational dofs is rather small. Hence, the importance of the corrected mass matrix will be larger for those modes in which the node rotations have a prevailing effect.

2.6.3.2 Effect of the Flexible Connections on the Seismic Response

We will examine next the influence of the connection flexibility in the response of a structure subjected to an earthquake ground motion modeled as a stationary stochastic process. The seis-

34

mic input is defined in terms of a set of ground response spectra. These spectra were obtained from an ensemble of 50 synthetically generated time histories. These sample time histories were generated from a broad-band Kanai-Tajimi spectral density function. The accelerograms were then modulated by envelope functions to reflect the build-up, strong motion and decay phases observed in recorded accelerograms. Figures (2.30) and (2.31) show the mean relative velocity and pseudo acceleration spectra obtained after statistically processing the ensemble of time history spectra. The response spectrum method was used to calculate the seismic design response. combining the modal responses according to a modified version of the SRSS rule [59]. Figure (2.32) shows the relative displacement response at three floor levels as a function of the flexibility of the connections defined in terms of fixity factors. The displacements were normalized by dividir, their values by those corresponding to the rigid connection case. In all the cases the response always increases with the flexibility of the connections. Also, for a given fixity factor, the relative importance of the flexible-connection effect increases with the floor number. The same observation holds for the elastic force response. The variation in the total shear force and the bending moment at the base of the structure are plotted in Figure (2.33). Comparing Figures (2.32) and (2.33), one can see that the joint flexibility has a more pronounced effect on the relative displacement response than on the elastic force response. It can be noticed that the rates of change of the variation of the displacement and moment responses increase notably when the fixity factors have a value less than 0.4.

Finally, the influence of the connection length on the seismic response is examined. Figures (2.34- 2.36) show the effect of the eccentricity ratio on the relative horizontal displacement at three different floor levels. The values of the displacements were normalized with respect to those corresponding to connections with negligible size. As expected since the structure becomes stiffer, the value of the displacements diminish as the length of the connections increases. This reduction in the response, however, is not significant: for a structure with rigid connections and eccentricity ratio of 0.1, the reduction is less than 15%. Comparing Figures (2.34- 2.36), one can see that the lower the floor level considered, the higher the influence of different values of the fixity factor on the connection-size effect. Figure (2.37) shows the influence of the eccentricity ratio on the design base shear. As the length of the connection increases, the base shear increases. This effect is more pronounced when the joints are rigid.

35
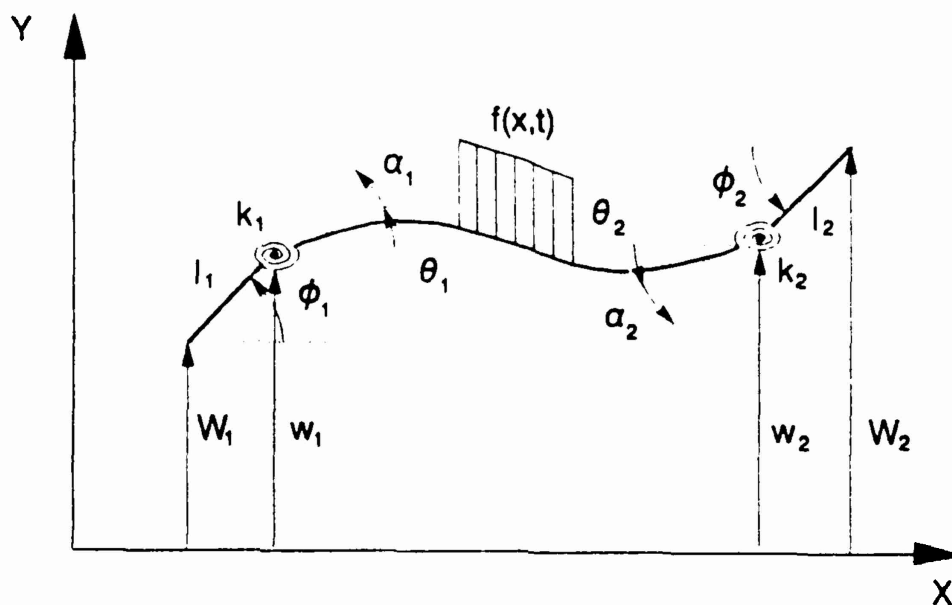
Figure 2.1: Beam Element with Flexible Joints and Connection Eccentricities.
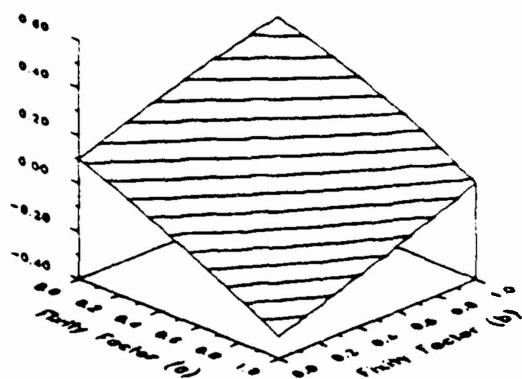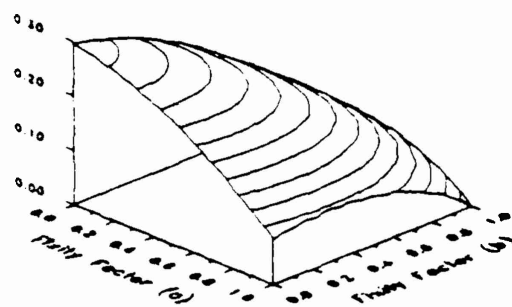
36

a) Coefficient ß1
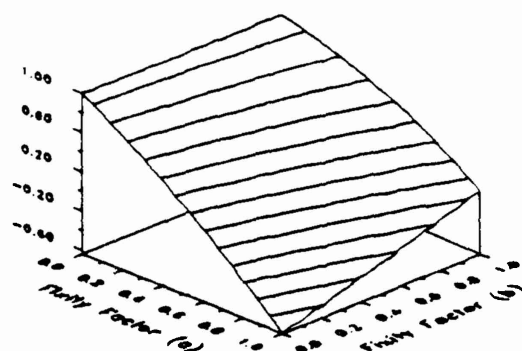
b) Coefficient ß2
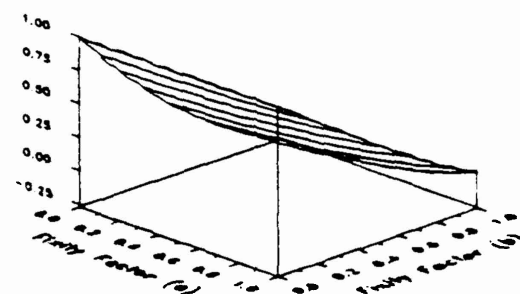
c) Coefficient ß3
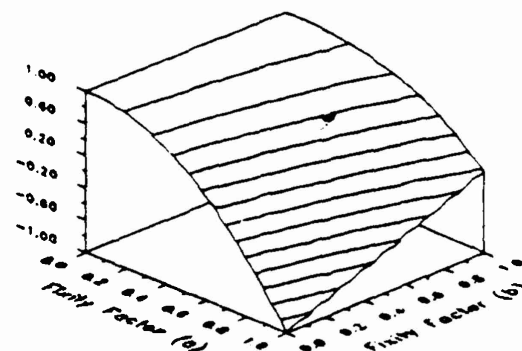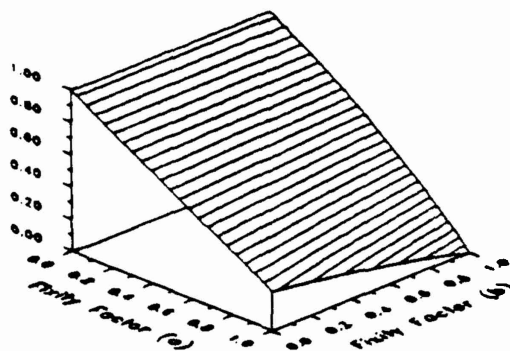
d) Coefficient ß4
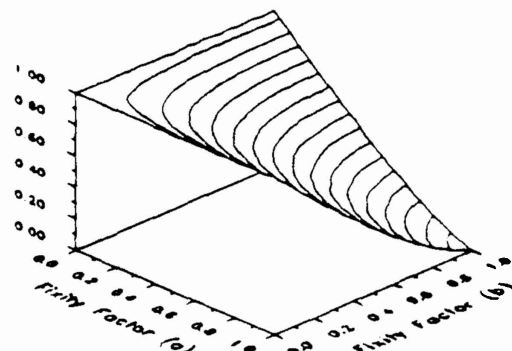
e) Coefficient ß5

f) Coefficient ß6

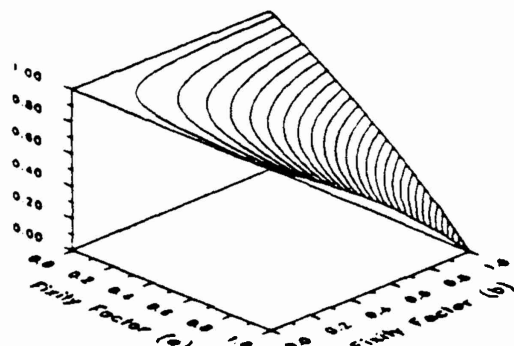Figure 2.2: Correction Coefficients for Mass Matrix.

37
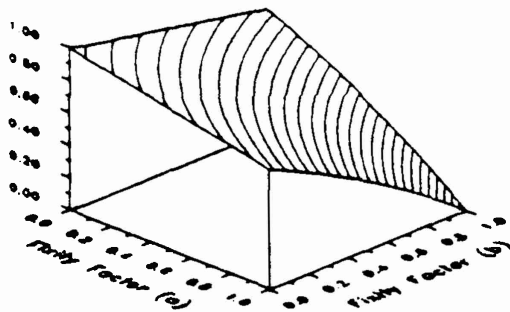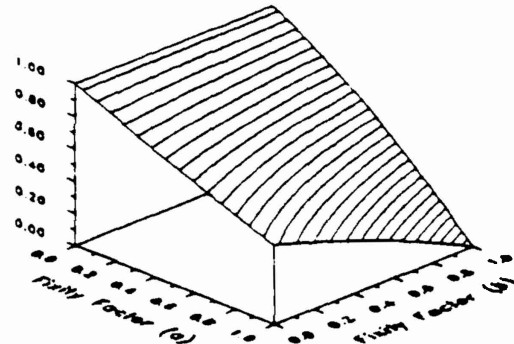
a) Coefficient $\alpha 1$

b) Coefficient $\alpha 2$

c) Coefficient $\alpha 3$

d) Coefficient $\alpha 4$

Figure 2.3: Correction Coefficients for Stiffness Matrix.
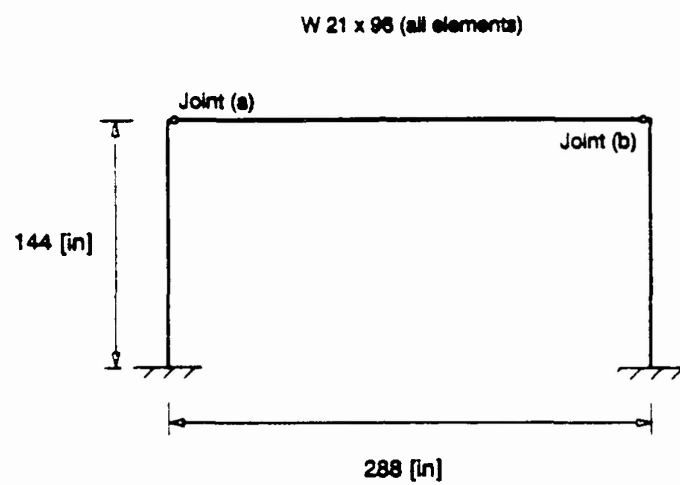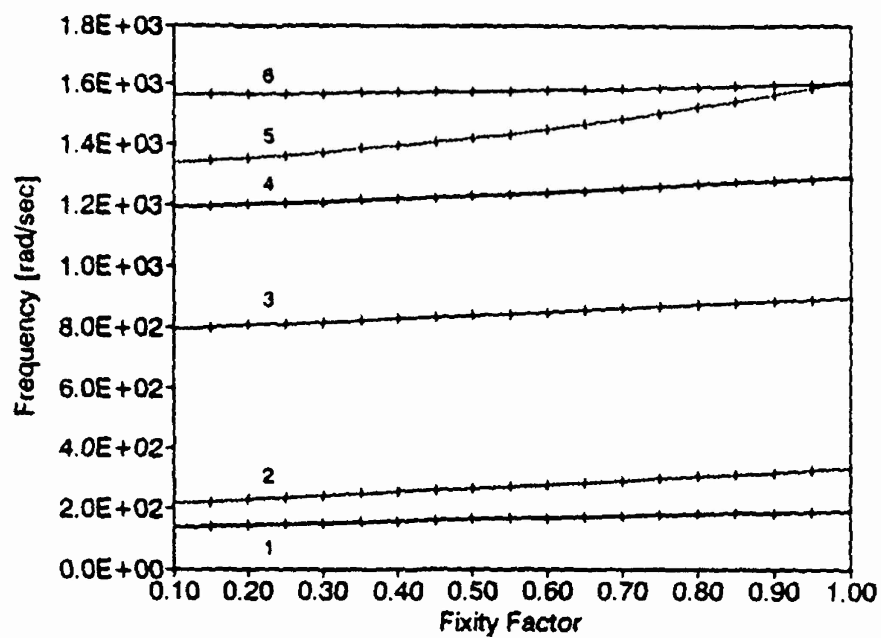
Figure 2.4: One-Story Frame for Numerical Examples.

Figure 2.5: Natural Frequencies as Function of Fixity Factors.
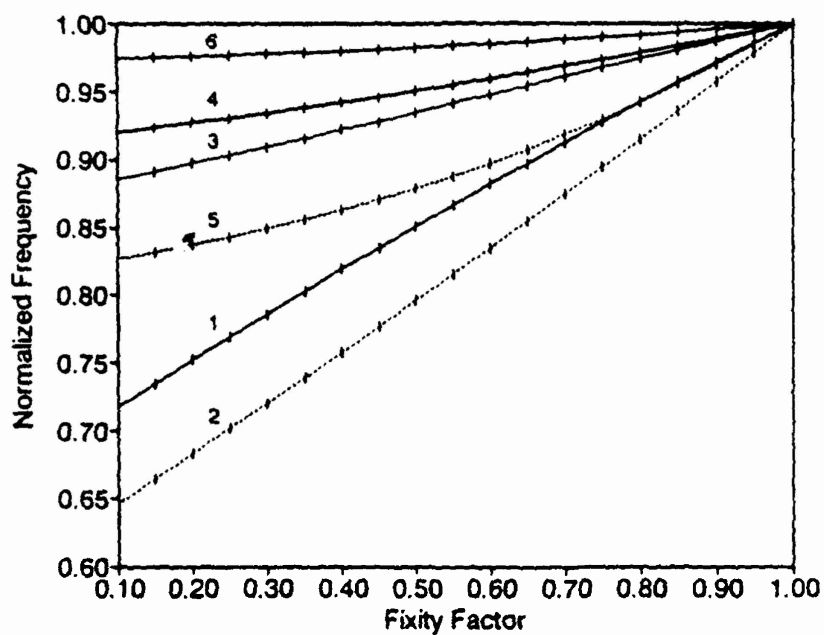
Figure 2.6: Normalized Natural Frequencies as Function of Fixity Factors.

40
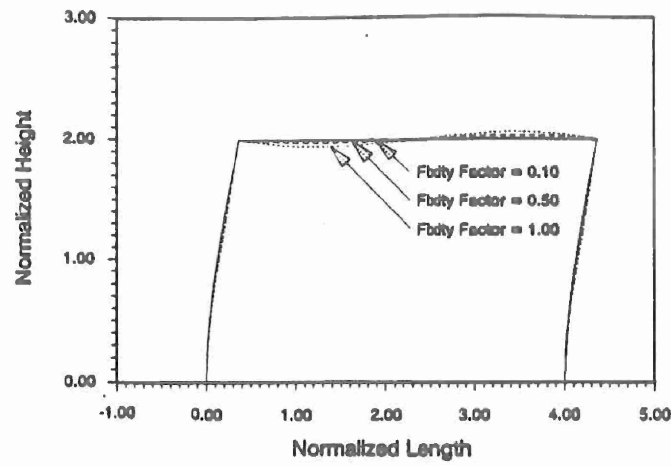
a) First Modal Shape

b) Second Modal Shape

c) Third Modal Shape

Figure 2.7: Scaled Modal Shapes of One-Story Frame.

41

Scaled Modal Shapes of One-Story Frame (cont.).
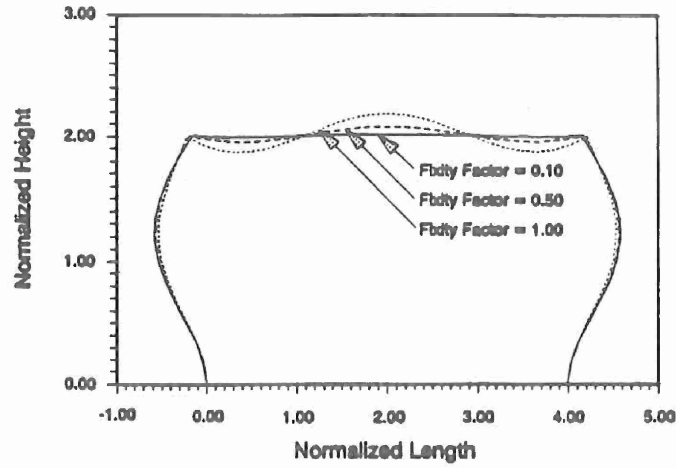
d) Fourth Modal Shape
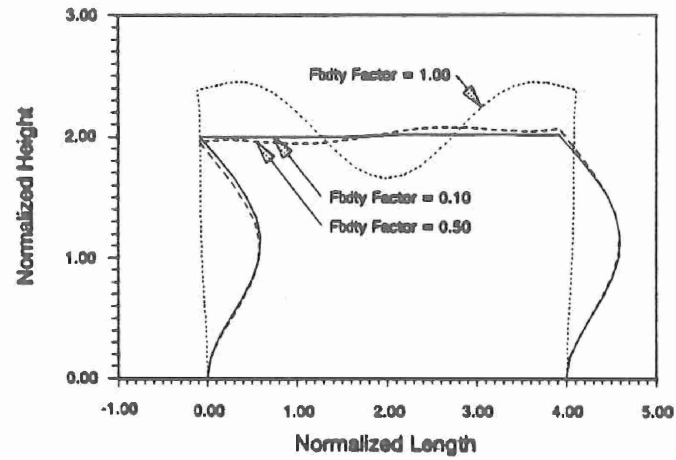
e) Fifth Modal Shape
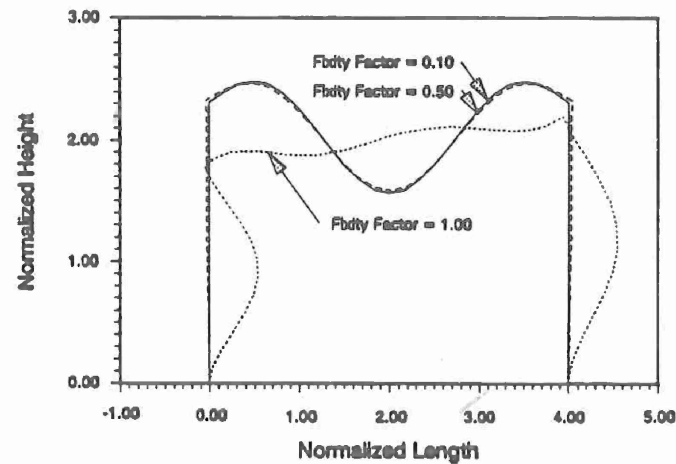
f) Sixth Modal Shape

42
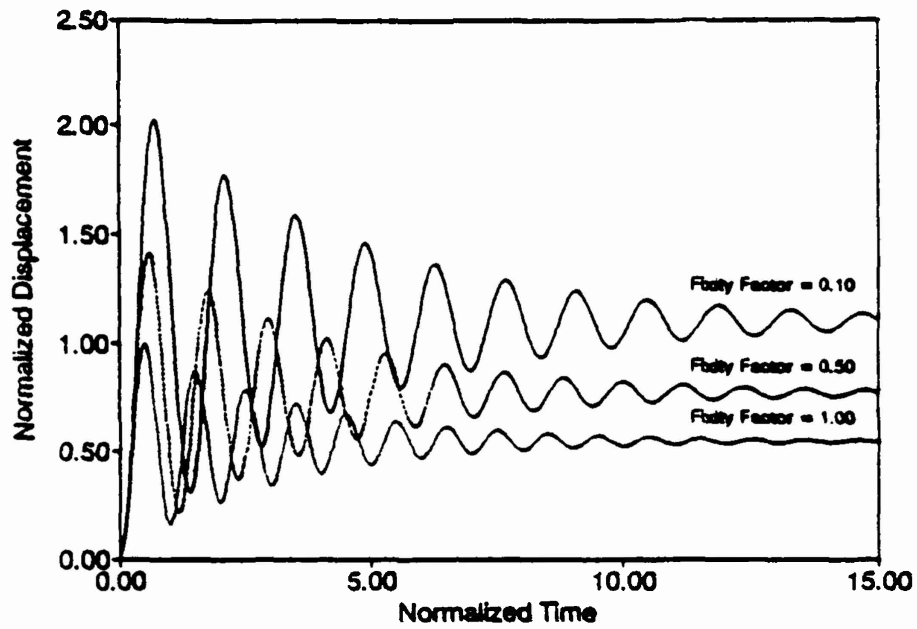
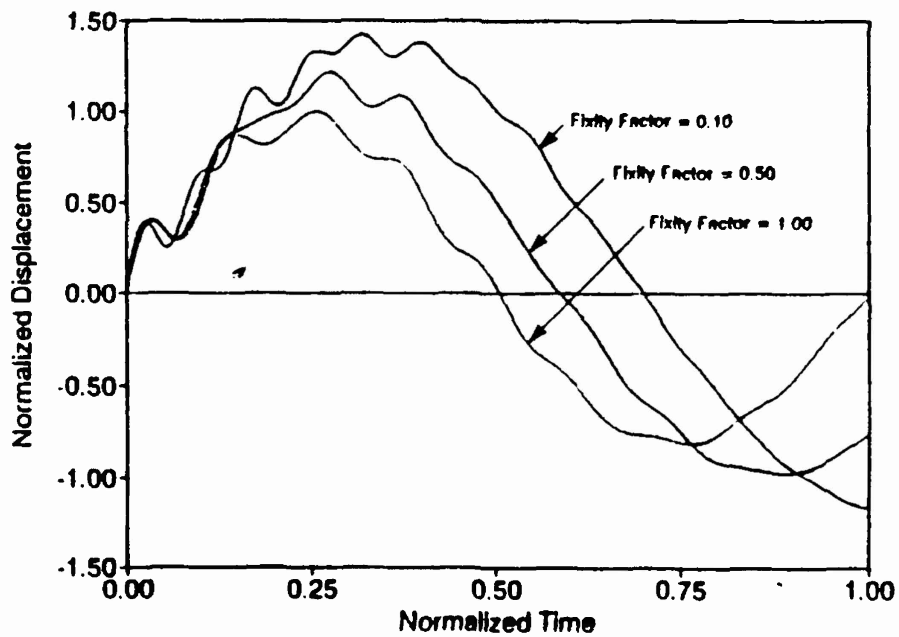Figure 2.8: Response to a Step Load: Horizontal Displacement of Joint (a).
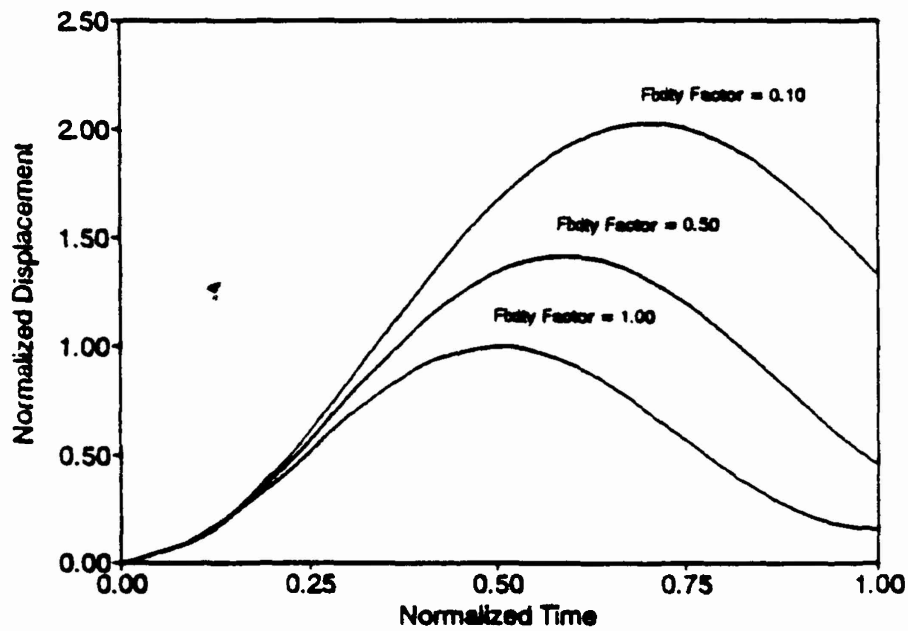
Figure 2.9: Response to a Step Load: Horizontal Displacement of Joint (a) (Initial Instants of Time).
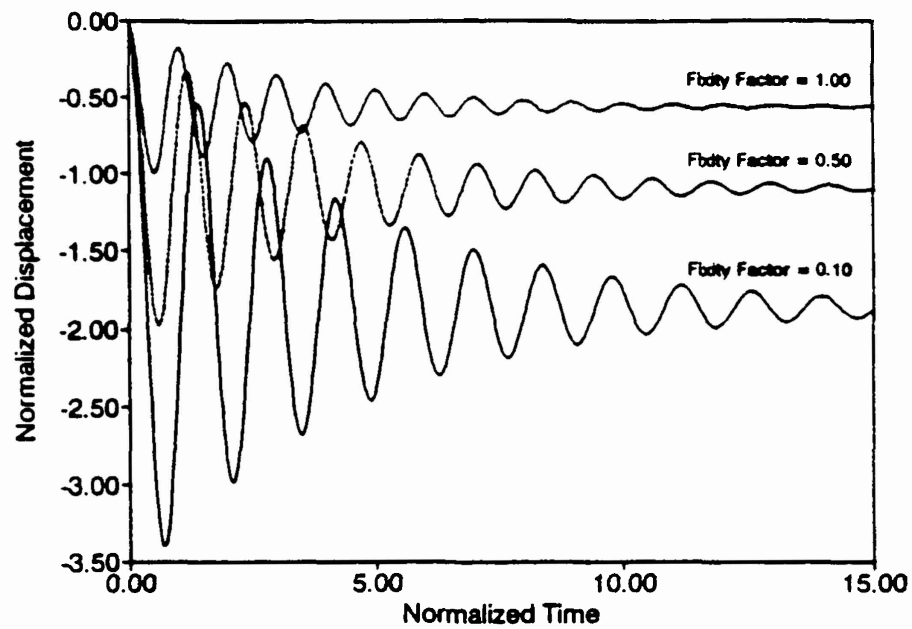
Figure 2.10: Response to a Step Load: Rotation of Joint (a).
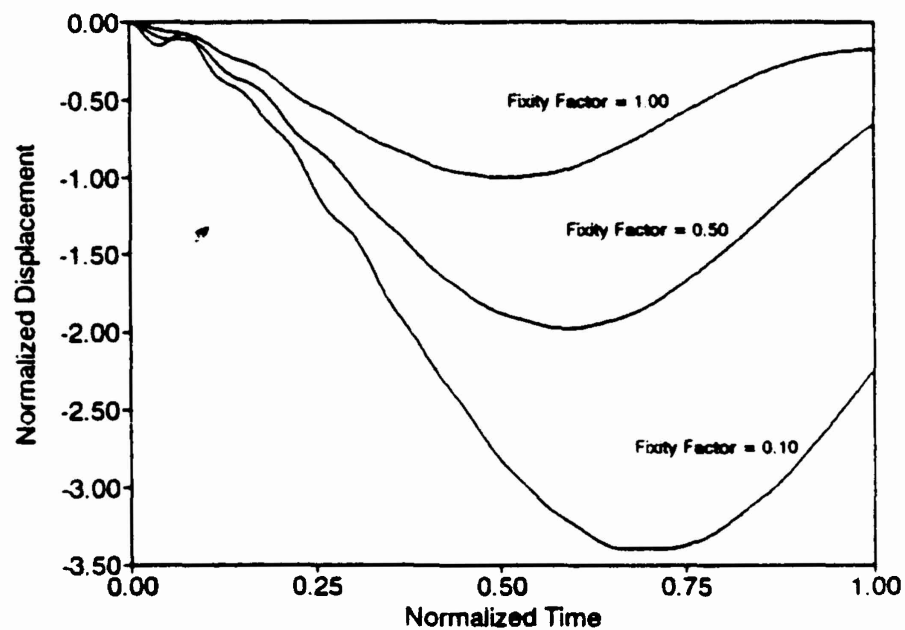
Figure 2.11: Response to a Step Load: Rotation of Joint (a) (Initial Instants of Time).

44
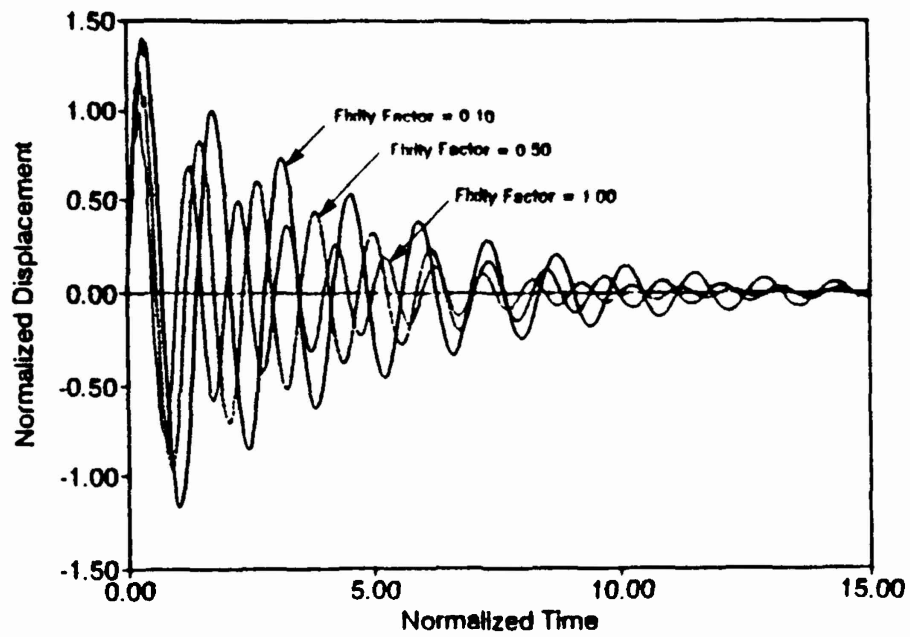
Figure 2.12: Response to an Impulse Load: Horizontal Displacement of Joint (a).

Figure 2.13: Respose to an Impulse Load: Horizontal Displacement of Joint (a) (Initial Instants of Time).

45
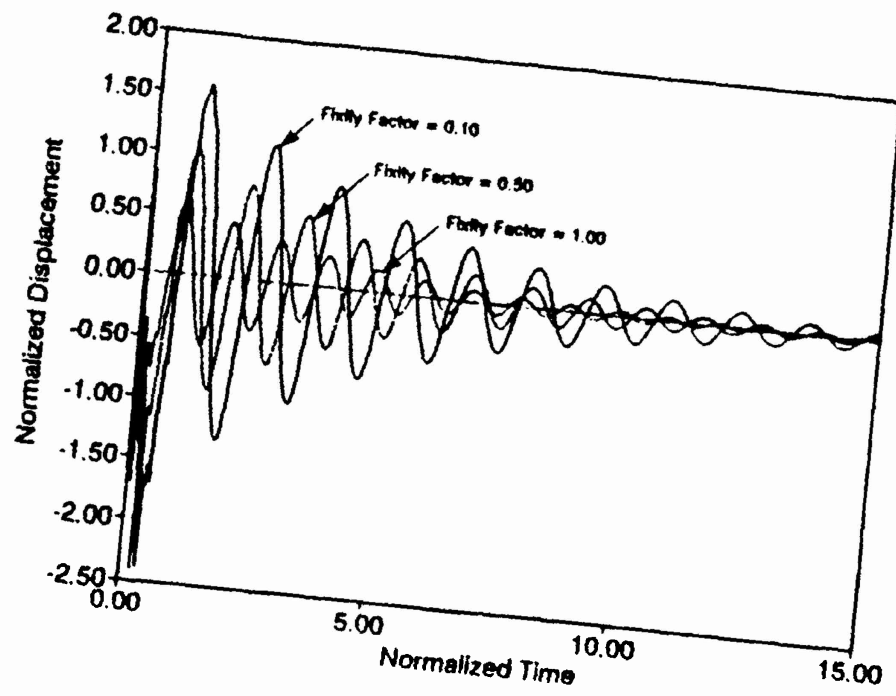
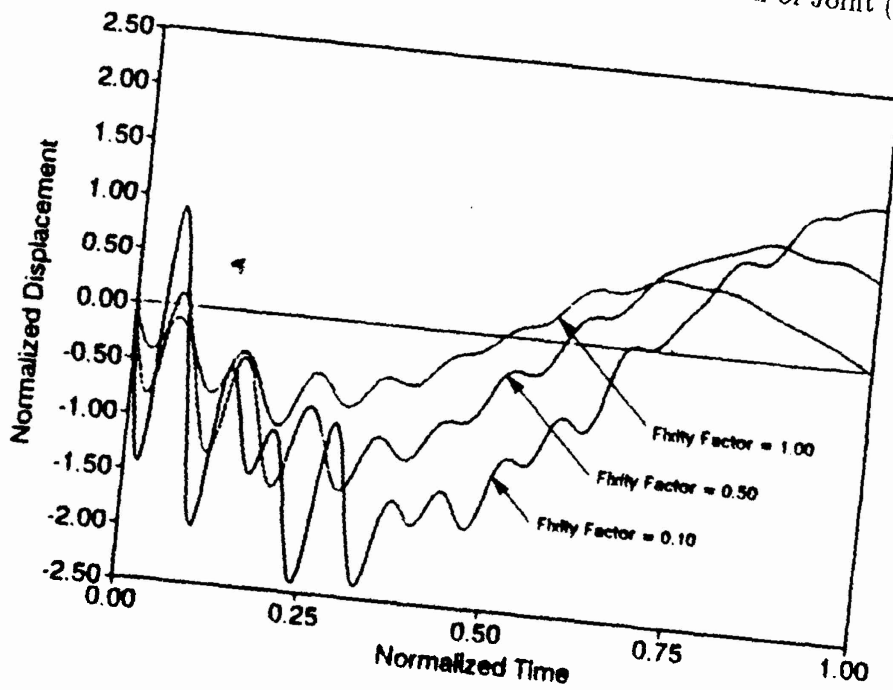Figure 2.14: Response to an Impulse Load: Rotation of Joint (a).

Figure 2.15: Response to an Impulse Load: Rotation of Joint (a) (Initial Instants of Time.
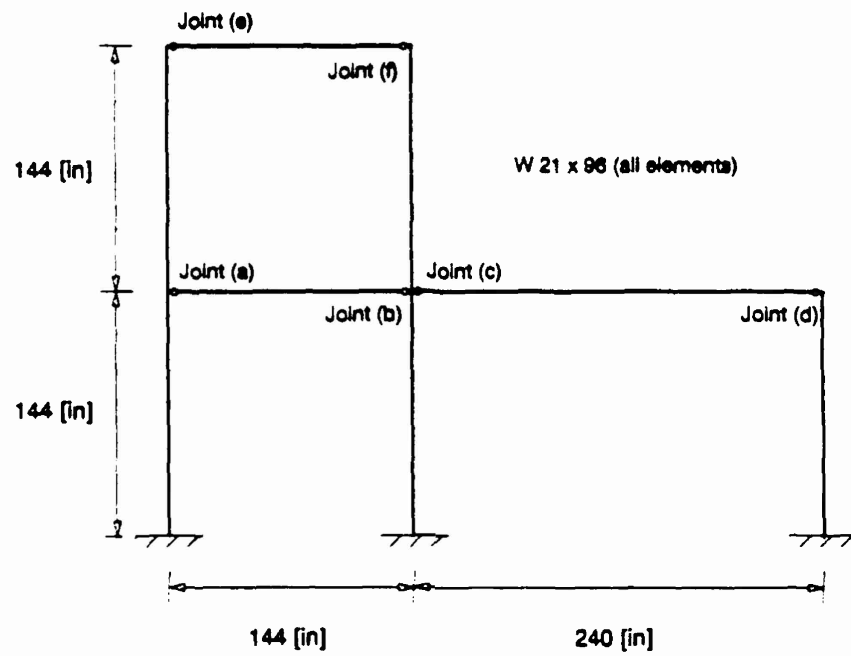
46

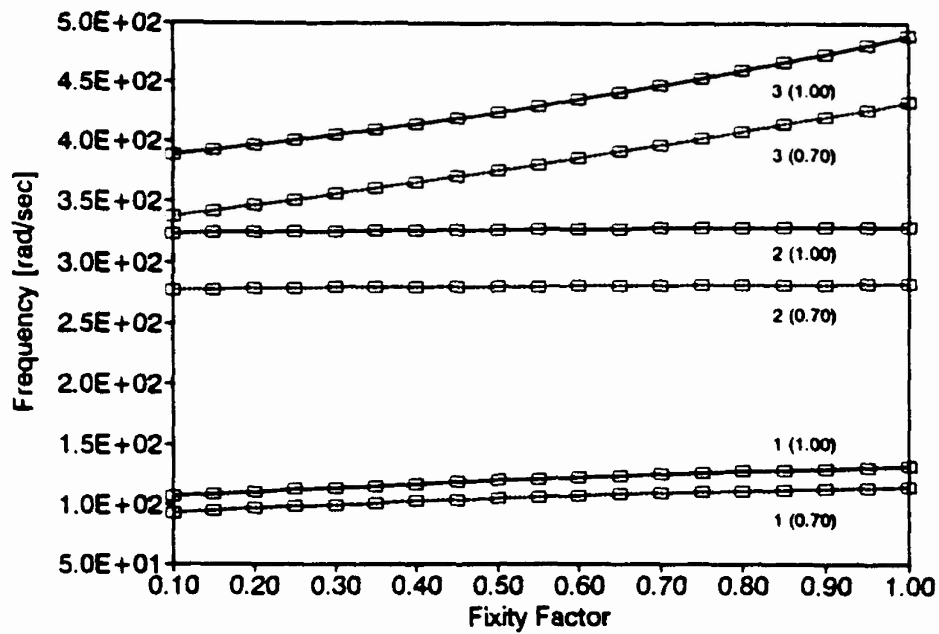Figure 2.16: Two-Story Frame for Numerical Examples.

47

Figure 2.17: Natural Frequencies as Function of Fixity Factors of Joints (a) and (c).
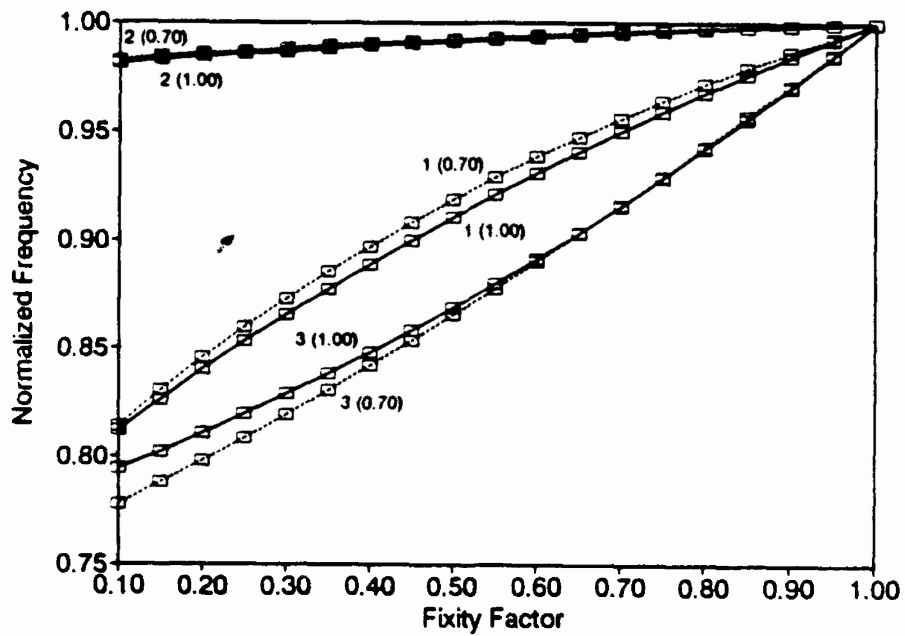

Figure 2.18: Normalized Natural Frequencies as Function of Fixity Factors of Joints (a) and (c).
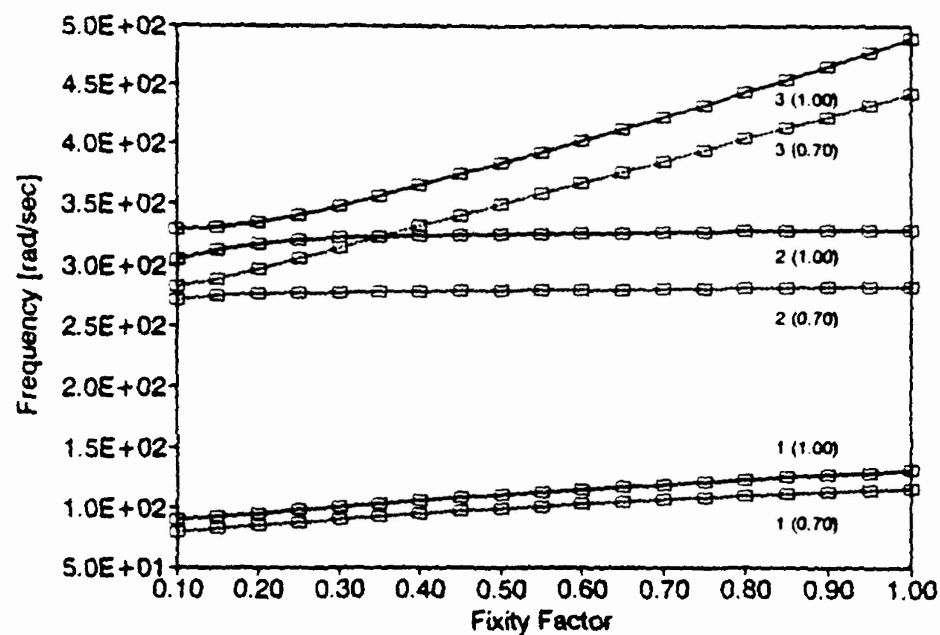
48

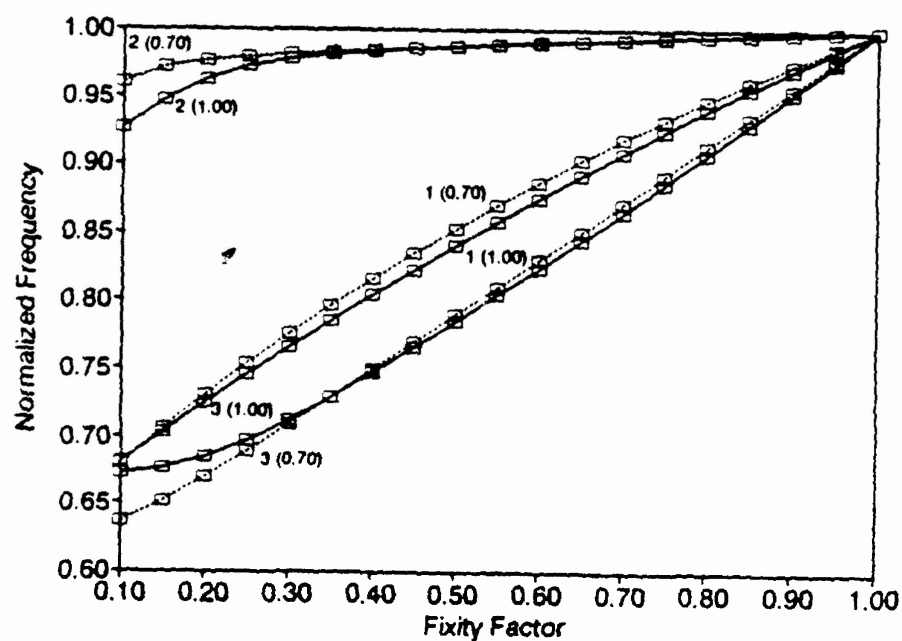Figure 2.19: Natural Frequencies as Function of Fixity Factors of Joints (a)-(d).

Figure 2.20: Normalized Natural Frequencies as Function of Fixity Factors of Joints (a)-(d).
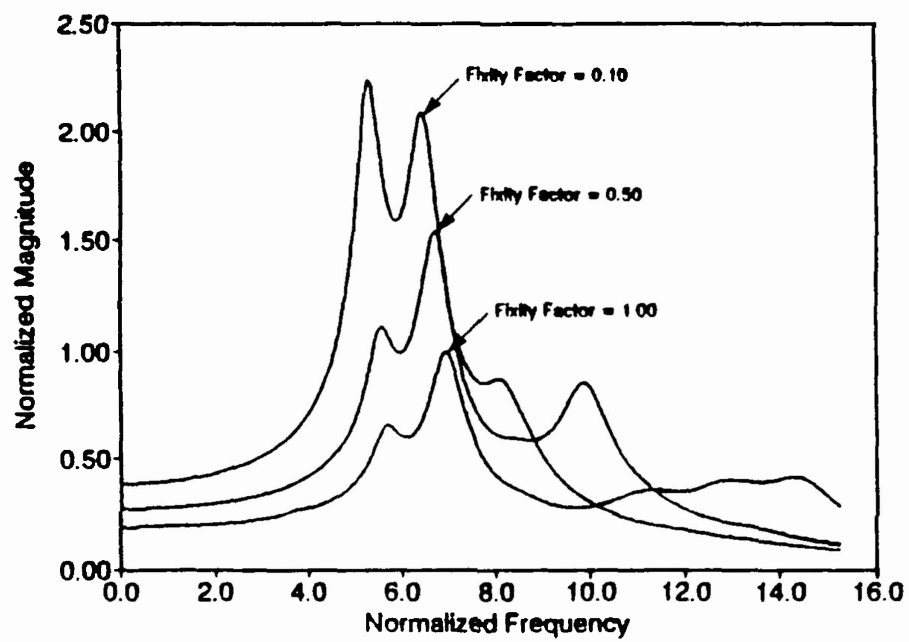
Figure 2.21: Magnitude of Frequency Response Function.
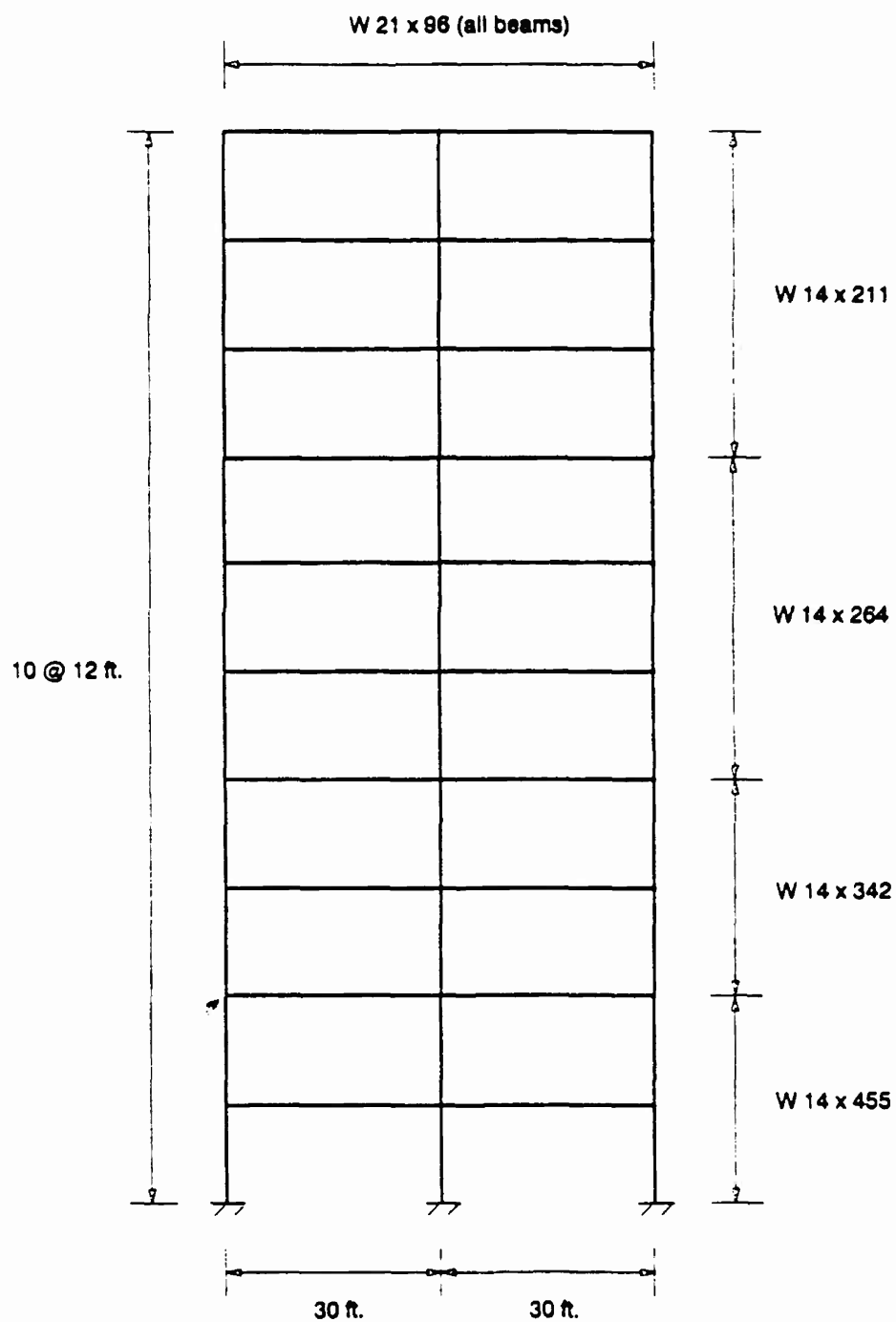
Figure 2.22: Ten-Story Frame for Numerical Examples.
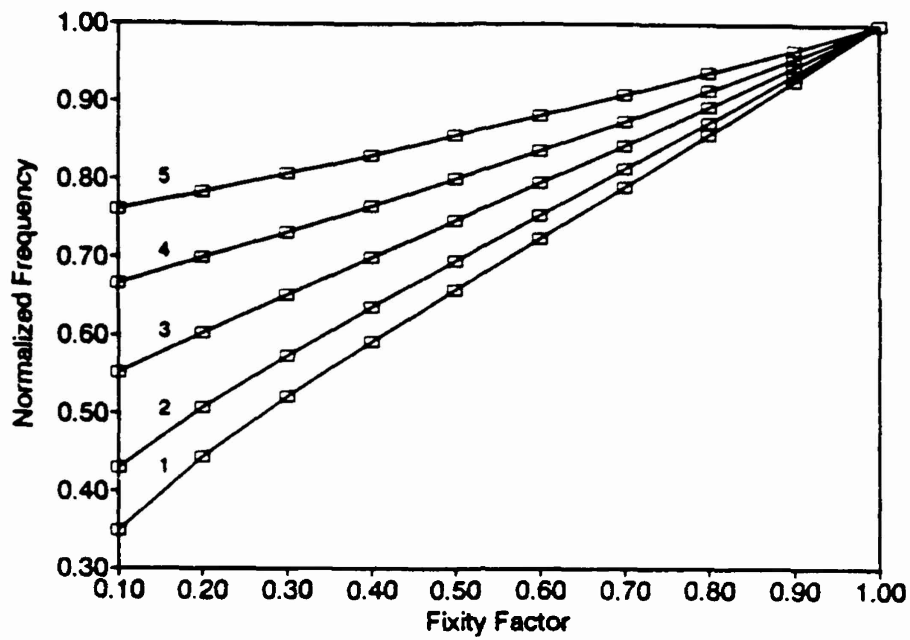
Figure 2.23: Normalized Natural Frequencies as a Function of Fixity Factors.
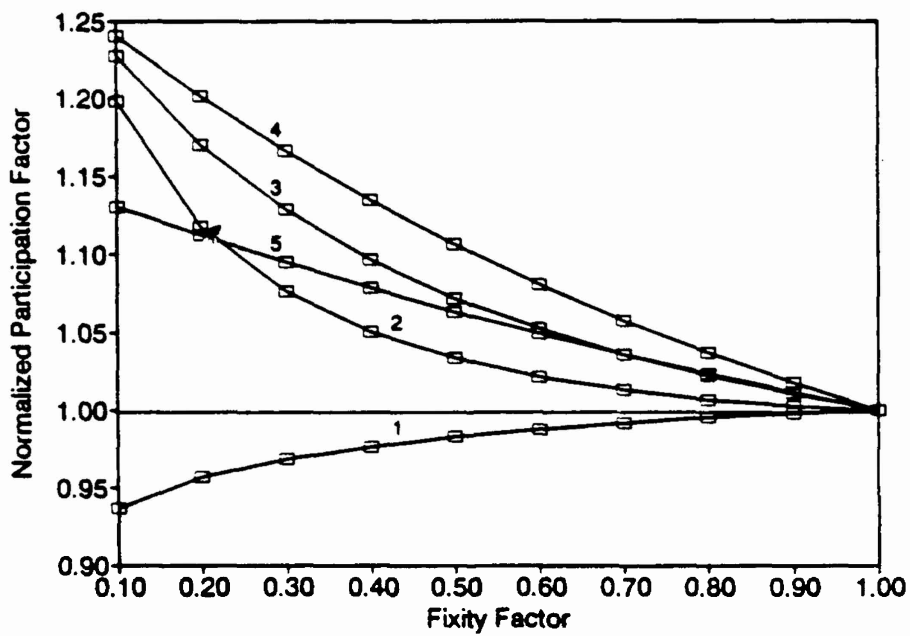
Figure 2.24: Normalized Participation Factors as a Function of Fixity Factors.
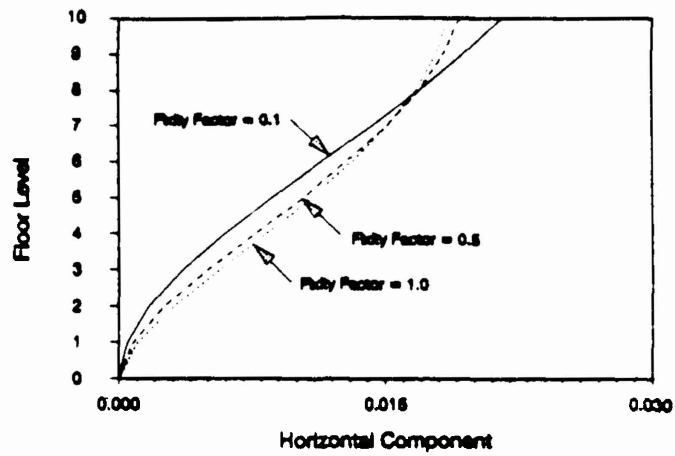
a) First Modal Shape
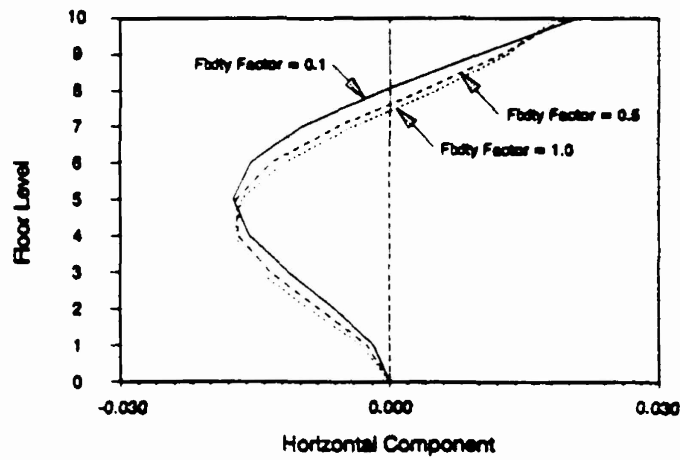
b) Second Modal Shape
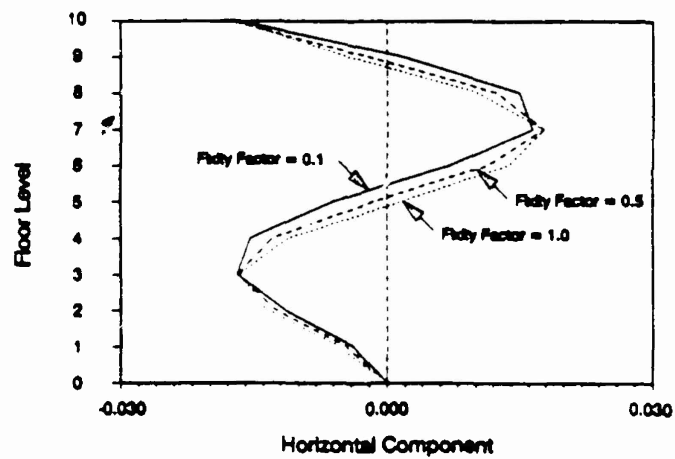
c) Third Modal Shape

Figure 2.25: Modal Shapes of Ten-Story Frame.

53
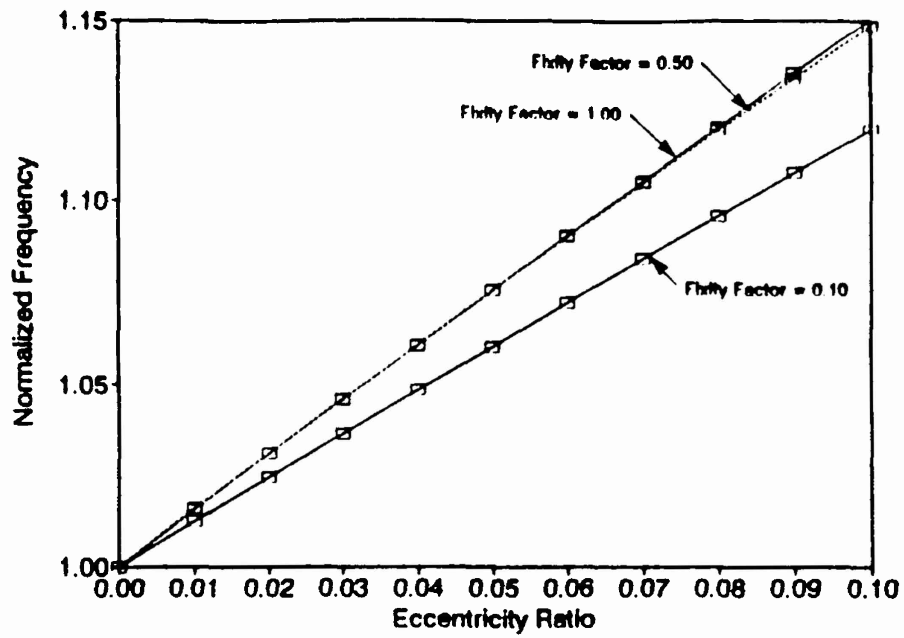
Figure 2.26: Normalized First Natural Frequency as Function of Eccentricity Ratios.
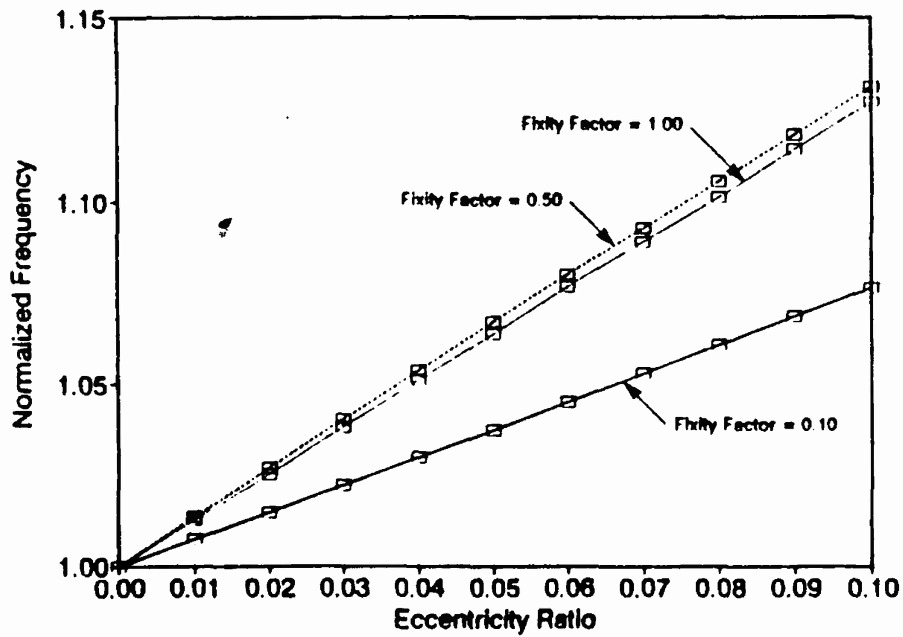
Figure 2.27: Normalized Second Natural Frequency as Function of Eccentricity Ratios.
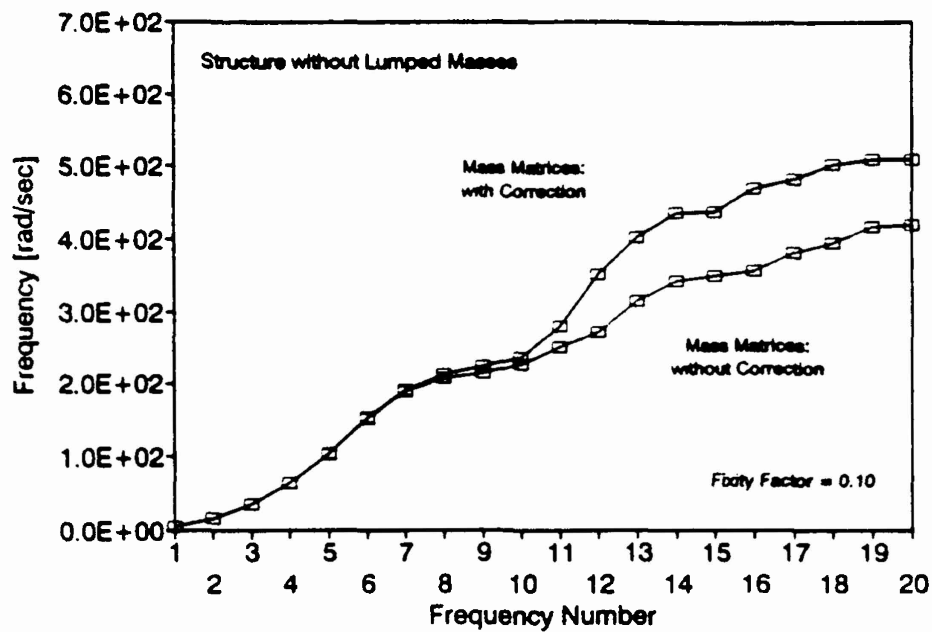
a) Fixity Factor = 0.10.
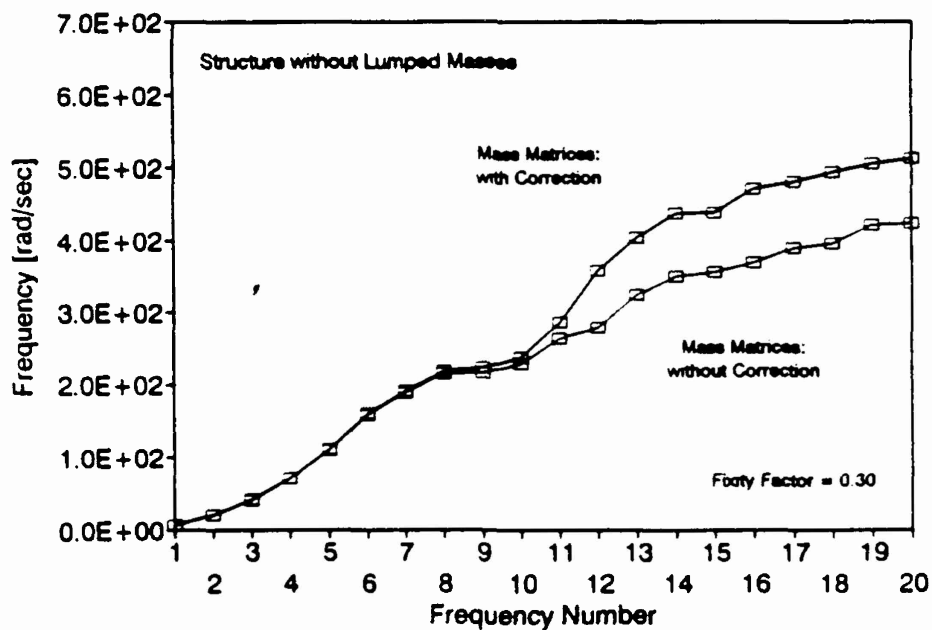
b) Fixity Factor = 0.30.

Figure 2.28: Effect of Mass Matrix Correction on Calculated Frequencies.

55
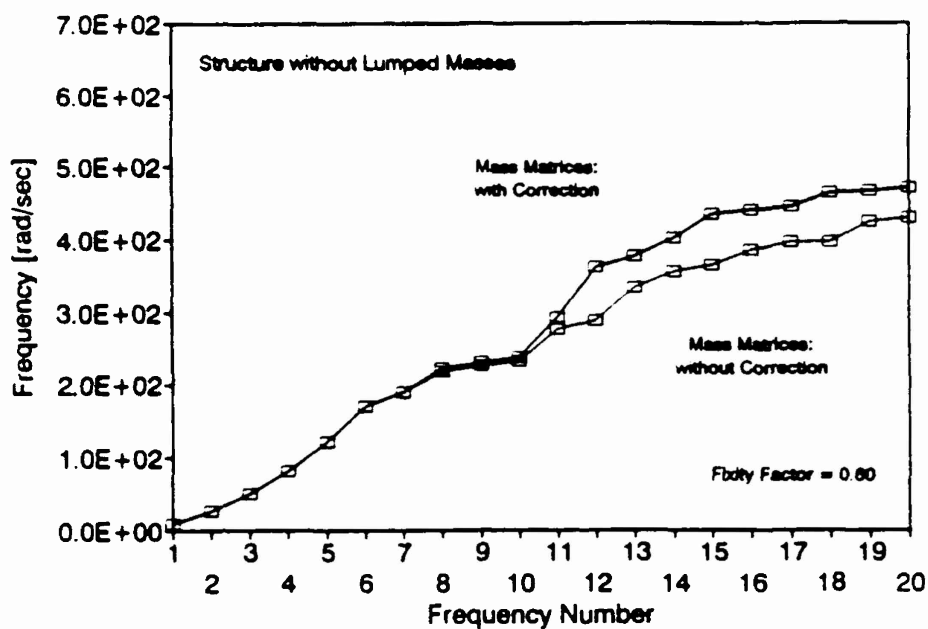
c) Fixity Factor = 0.60.
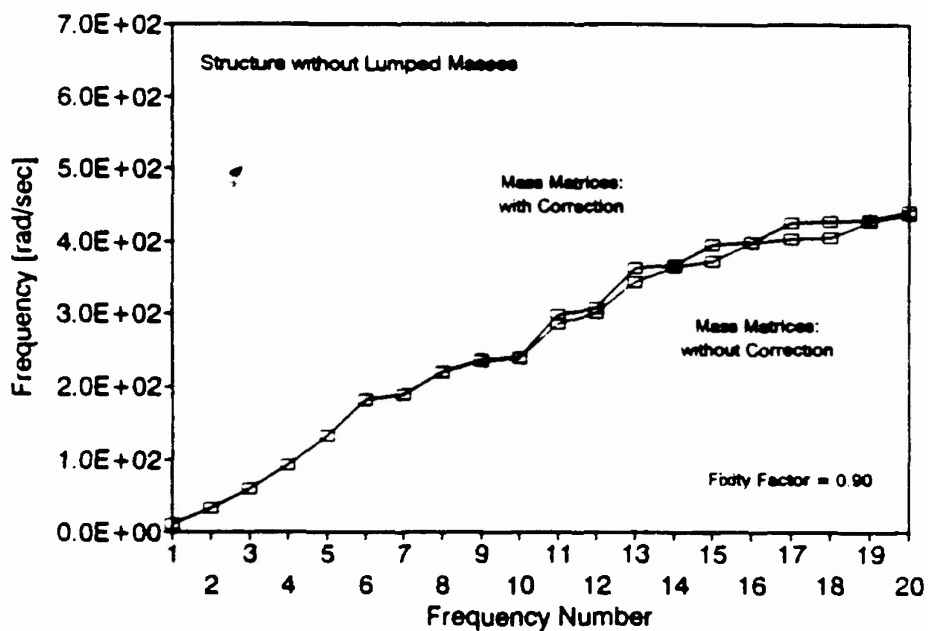
d) Fixity Factor = 0.90.
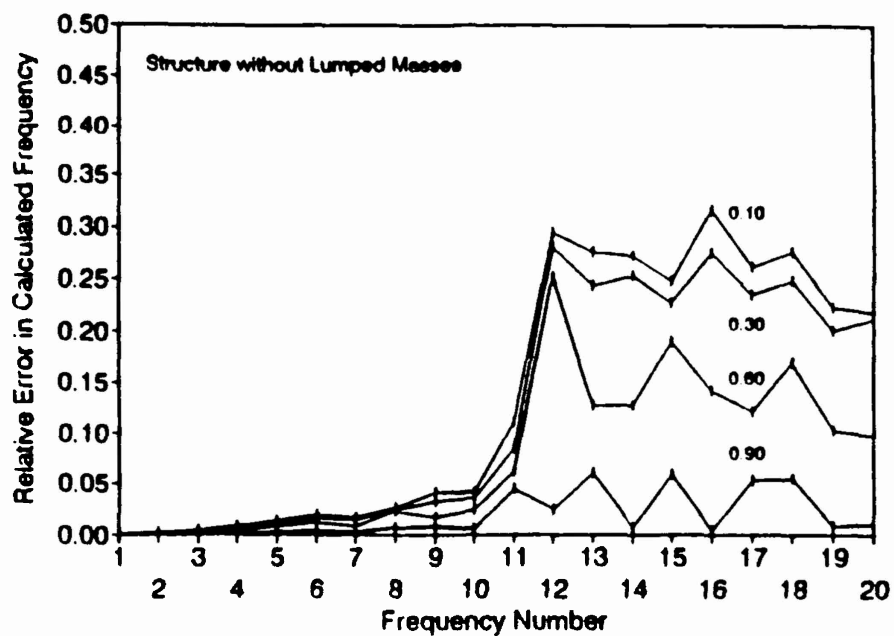
Figure 2.29: Relative Error in Natural Frequencies.
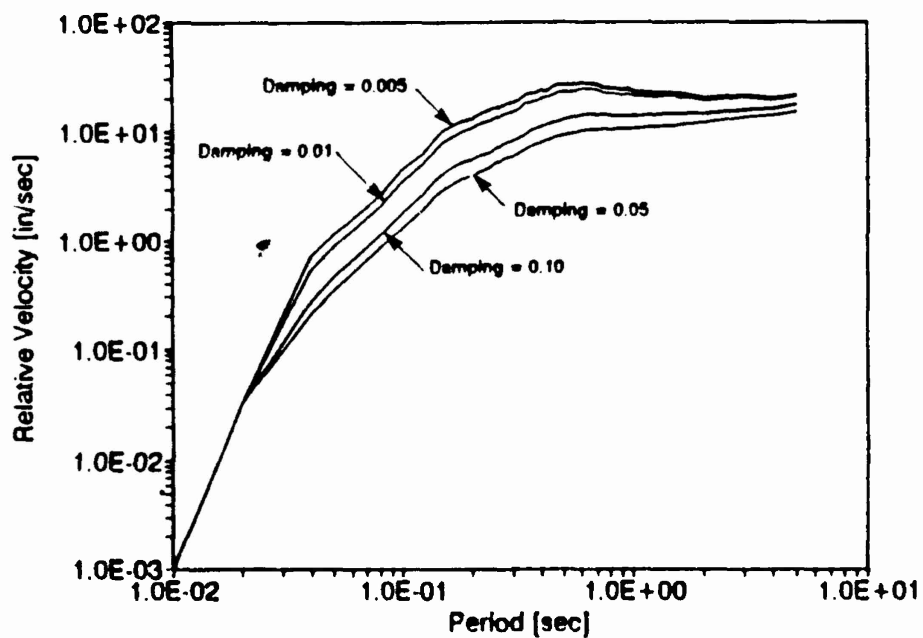
Figure 2.30: Relative Velocity Ground Response Spectra.

57
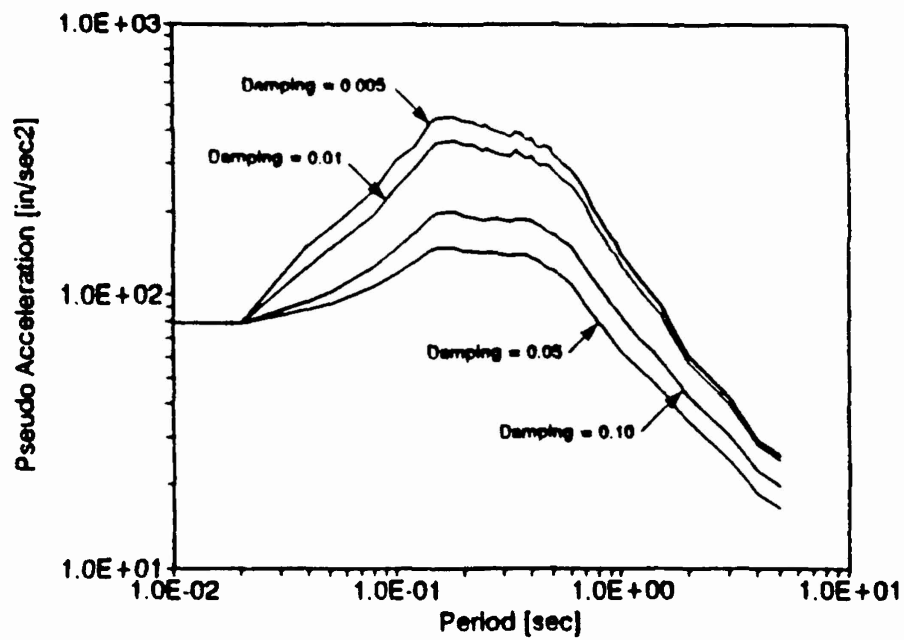
Figure 2.31: Pseudo Acceleration Ground Response Spectra
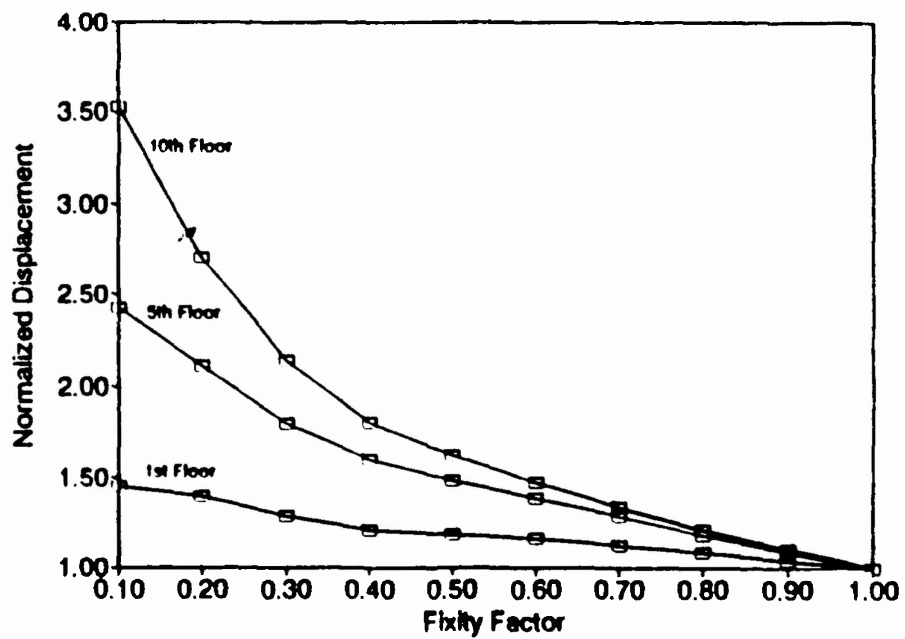
Figure 2.32: Normalized Design Displacement as Function of Fixity Factors.

58
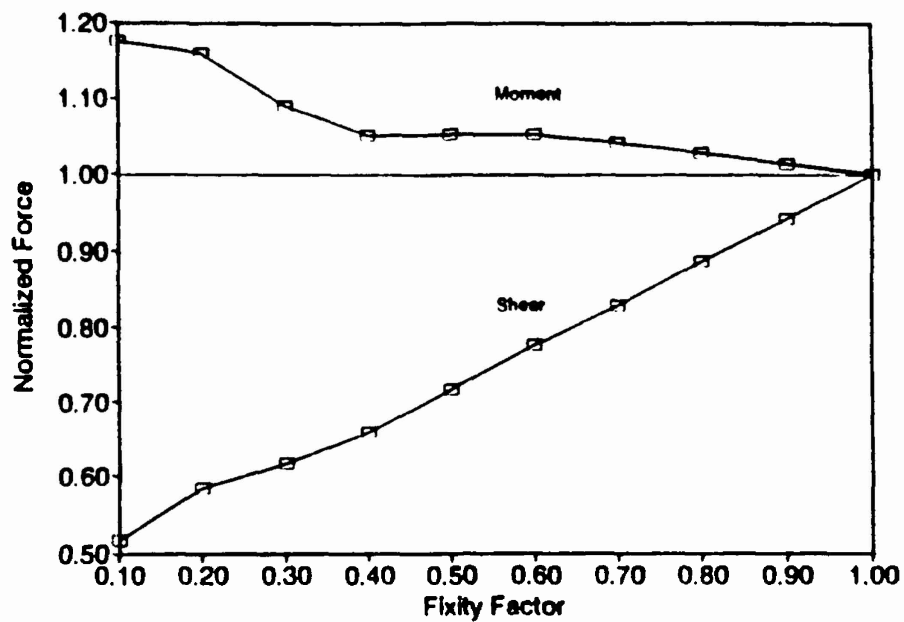
Figure 2.33: Normalized Design Forces as Function of Fixity Factors.
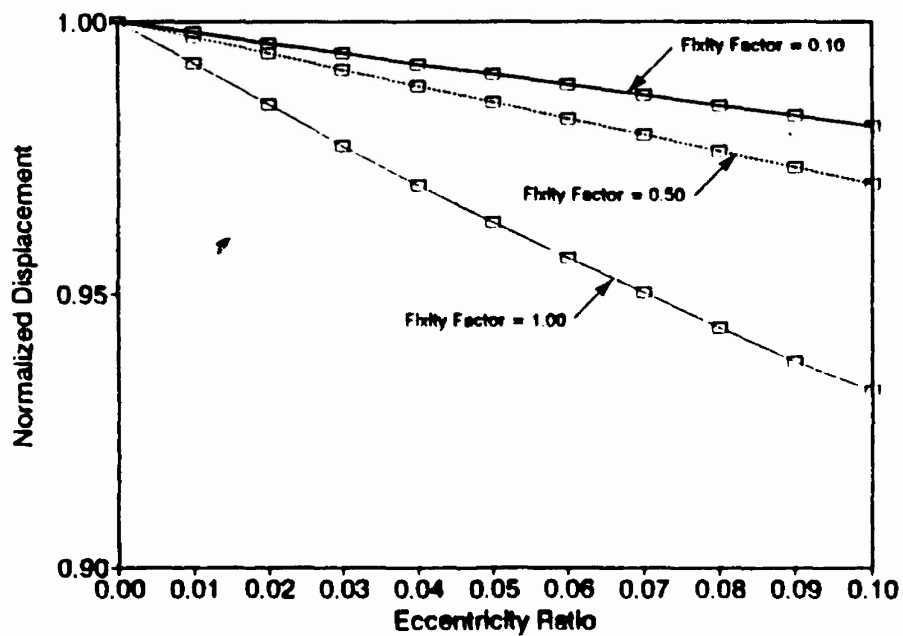
Figure 2.34: Normalized Design Displacement of First Floor as Function of Eccentricity Ratios.

59
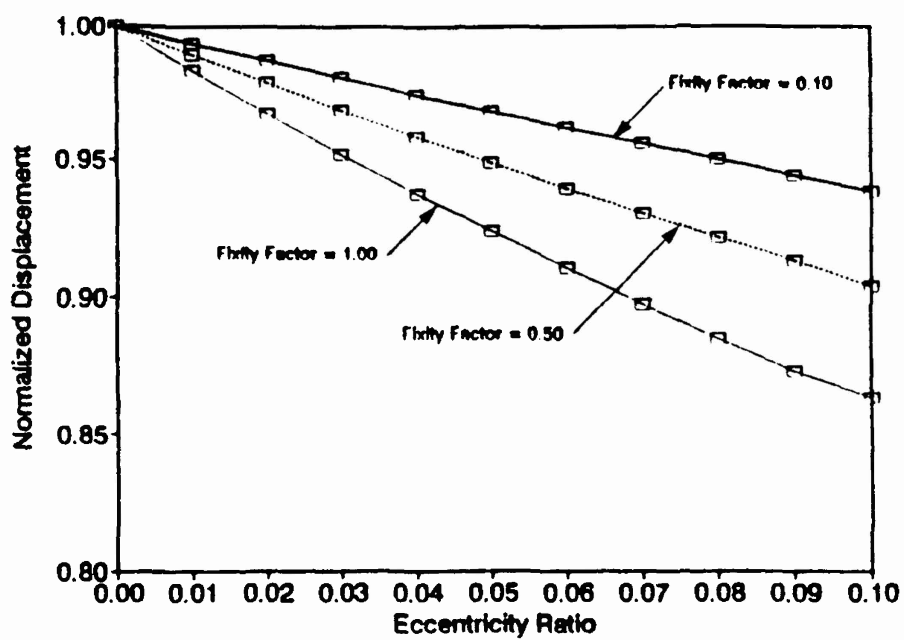
Figure 2.35: Normalized Design Displacement of Second Floor as Function of Eccentricity Ratios.
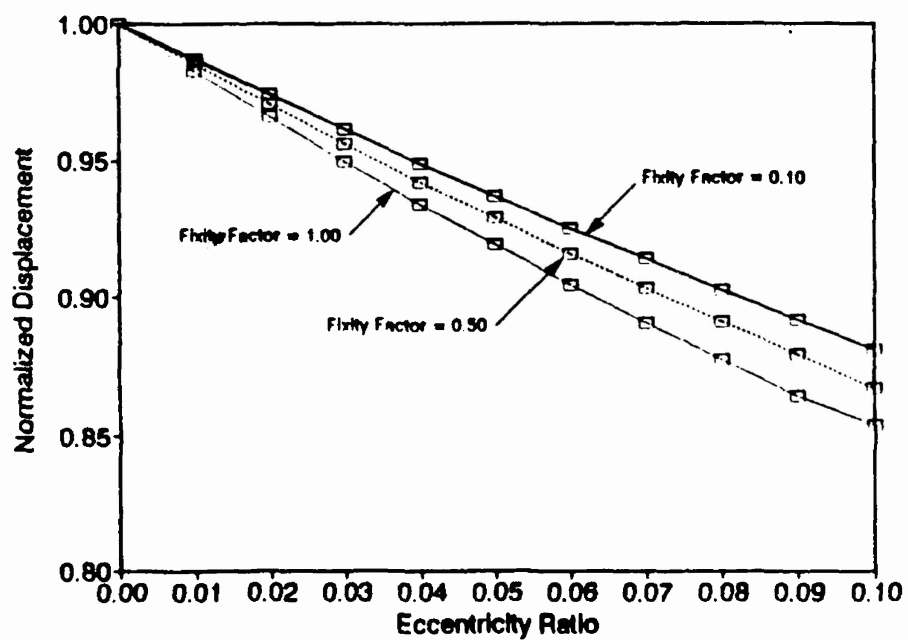
Figure 2.36: Normalized Design Displacement of Third Floor as Function of Eccentricity Ratios.
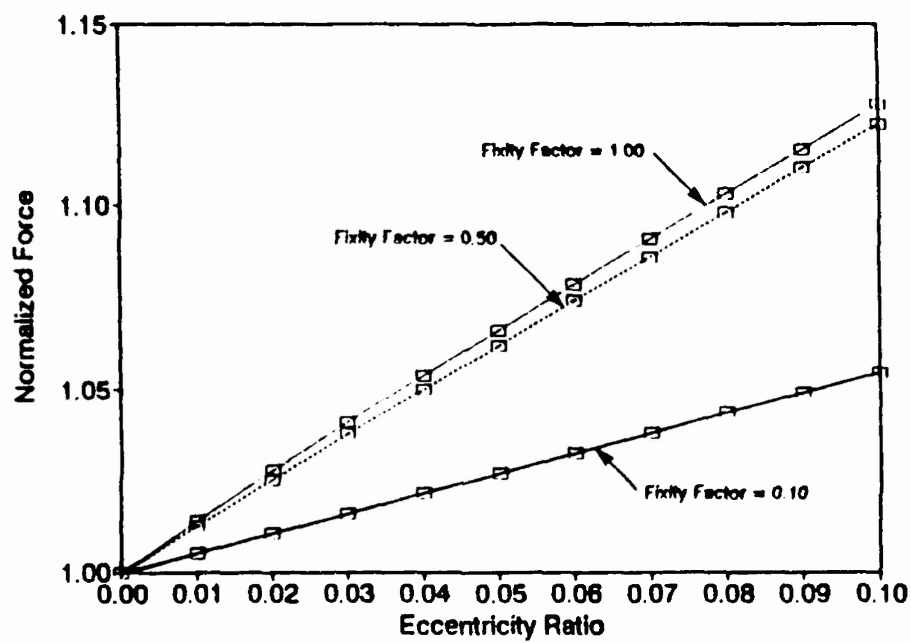
Figure 2.37: Normalized Design Base Shear as Function of Eccentricity Ratios.

Chapter 3

Sensitivity Analysis of Structural Eigenproperties

3.1 Introduction

It was previously mentioned that one of the goals of the present research is to characterize how the response of structures whose connections are not perfectly rigid is modified by changes or perturbations in the values of the stiffness coefficients of the connections. One approach to solve this problem is by means of a sensitivity analysis of the structural eigenproperties with respect to perturbations in the values of the fixity factors.

Sensitivity analysis was first used to asses the effect of varying parameters in the mathematical models of control systems. More recently, interest in automated structural optimization led to the development of methods to calculate the derivatives of structural response quantities. such as stresses, displacements. buckling loads and natural frequencies. The calculation of these derivatives with respect to the design variables (cross-sectional areas, widths, thicknesses, etc.) is required in all the optimization algorithms and it is the dominant contribution to the total cost of the optimization process. In addition, structural sensitivity analysis was also applied in approximate analyses, improvement of analytical models, and to study the influence of design changes without a trial and error process.

In this chapter, we will obtain closed-form expressions to calculate the derivatives of the eigenvalues and eigenvectors with respect to the fixity factors associated with the stiffness of the connections. This sensitivity information, besides providing the tools to understand and predict how the changes in the design variables (the connection stiffnesses) affect the dynamic behavior of the structure, will be the starting point in the calculation of the statistics of the eigenproperties when the stiffness values are regarded as random variables.

3.2 Derivatives of Eigenvalues and Eigenvectors

A central step in the calculation of the response of discrete- parameter multi-degree-of-freedom systems by modal superposition is the solution of the algebraic eigenvalue problem:

$$\left\{ [K] - \omega_i^2 [M] \right\} \phi_i = 0 \quad ; \quad i = 1, 2, \ldots, N \tag{3.1}$$

where $[M]$ is the global mass matrix, $[K]$ is the global stiffness matrix, and ω_i is the $i-th$ natural frequency of vibration associated with the $i-th$ modal shape vector ϕ_i. The total number of unconstrained degrees of freedom of the system is N.

The global structural matrices are obtained by assembling the contributions of the finite element matrices in the usual way. For a structure modeled as an assemblage of frame elements with flexible connections, these matrices depend on the fixity factors of each element. To study the effect produced by changes in the stiffness values of some of the connections, the eigenvalue problem will be written as a function of the vector of fixity factors μ:

$$\{[K(\mu)] - \lambda_i(\mu)[M(\mu)]\}\,\phi_i(\mu) = 0 \quad ; \quad i = 1, 2, ..., N \tag{3.2}$$

where $\lambda_i = \omega_i{}^2$ is the $i-th$ eigenvalue and:

$$\mu = \{\mu_1 \ \mu_2 \ \cdots \ \mu_R\}^T \tag{3.3}$$

The subscript R represents the number of fixity factors that are considered to be the variables of the problem, i.e., the number of flexible connections.

The eigenvectors ϕ_i associated with the eigenvalue problem in equation (3.1) will be normalized with respect to the global mass matrix:

$$\phi_i(\mu)^T\,[M(\mu)]\,\phi_i(\mu) = 1 \tag{3.4}$$

The changes in the global matrices produced by a finite perturbation of the variables from a set of initial values can be expressed in the form of a Taylor series expansion as follows:

$$[K(\mu)] = [\bar{K}] + \sum_{m=1}^{R}\left[K_m^I\right]\delta_m + \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\left[K_{mn}^{II}\right]\delta_m\,\delta_n \tag{3.5}$$

$$[M(\mu)] = [\bar{M}] + \sum_{m=1}^{R}\left[M_m^I\right]\delta_m + \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\left[M_{mn}^{II}\right]\delta_m\,\delta_n \tag{3.6}$$

where the series is truncated at the second order. The following notation was introduced in equations (3.5) and (3.6). The variable δ_i represents the change of each fixity factor from the initial state,

$$\delta_i = \mu_i - \bar{\mu}_i \tag{3.7}$$

where $\bar{\mu}_i$ the initial value of the $i-th$ fixity factor. The system matrices evaluated at the initial state are denoted by:

$$[\bar{K}] = [K(\bar{\mu})] \quad ; \quad [\bar{M}] = [M(\bar{\mu})] \tag{3.8}$$

and the superscripts I and II denote, respectively, the first and second order rate of change of the system matrices, which are obtained as the derivatives of the system matrices evaluated at the initial state vector $\bar{\mu}$:

63

$$\left[K_m^I\right] = \left.\frac{\partial\left[K\left(\mu\right)\right]}{\partial\mu_m}\right|_{\mu=\bar{\mu}} \quad : \quad \left[M_m^I\right] = \left.\frac{\partial\left[M\left(\mu\right)\right]}{\partial\mu_m}\right|_{\mu=\bar{\mu}} \tag{3.9}$$

$$\left[K_{mn}^{II}\right] = \left.\frac{\partial^2\left[K\left(\mu\right)\right]}{\partial\mu_m\,\partial\mu_n}\right|_{\mu=\bar{\mu}} \quad : \quad \left[M_{mn}^{II}\right] = \left.\frac{\partial^2\left[M\left(\mu\right)\right]}{\partial\mu_m\,\partial\mu_n}\right|_{\mu=\bar{\mu}} \tag{3.10}$$

In order to obtain more condensed expressions, the following matrices are introduced:

$$[K_1] = \sum_{m=1}^{R}\left[K_m^I\right]\delta_m \quad : \quad [M_1] = \sum_{m=1}^{R}\left[M_m^I\right]\delta_m \tag{3.11}$$

$$[K_2] = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\left[K_{mn}^{II}\right] \quad : \quad [M_2] = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R}\left[M_{mn}^{II}\right] \tag{3.12}$$

Equations (3.5) and (3.6) can now be written as:

$$[K\left(\bar{\mu}\right)] = [\bar{K}] + \varepsilon[K_1] + \varepsilon^2[K_2] \tag{3.13}$$

$$[M\left(\bar{\mu}\right)] = [\bar{M}] + \varepsilon[M_1] + \varepsilon^2[M_2] \tag{3.14}$$

The changes in the system matrices from a initial state are expressed in terms of a perturbation parameter ε which is a book-keeping device that will allow us to keep track of the order of the terms in the expansions. The subscripts 1 and 2 identify, respectively, the first and second order contributions to the perturbed matrices.

Akin to the previous representation of the changes in the system matrices, the changes in the eigenvalues and eigenvectors are defined through expansions which will be truncated at the second order level:

$$\lambda_i\left(\mu\right) = \bar{\lambda}_i + \sum_{m=1}^{R}\lambda_{im}^I\,\delta_m + \sum_{m=1}^{R}\sum_{n=1}^{R}\lambda_{imn}^{II}\,\delta_m\,\delta_n \tag{3.15}$$

$$\phi_i\left(\mu\right) = \bar{\phi}_i + \sum_{m=1}^{R}\phi_{im}^I\,\delta_m + \sum_{m=1}^{R}\sum_{n=1}^{R}\phi_{imn}^{II}\,\delta_m\,\delta_n \tag{3.16}$$

The rates of change of the eigenvalues and eigenvectors are defined similarly to those of the system matrices given in equations (3.9) and (3.10), that is:

$$\lambda_{im}^I = \left.\frac{\partial\lambda_i}{\partial\mu_m}\right|_{\mu=\bar{\mu}} \quad ; \quad \phi_{im}^I = \left.\frac{\partial\phi_i}{\partial\mu_m}\right|_{\mu=\bar{\mu}} \tag{3.17}$$

$$\lambda_{imn}^{II} = \left.\frac{\partial^2\lambda_i}{\partial\mu_m\,\partial\mu_n}\right|_{\mu=\bar{\mu}} \quad ; \quad \phi_{imn}^{II} = \left.\frac{\partial^2\phi_i}{\partial\mu_m\,\partial\mu_n}\right|_{\mu=\bar{\mu}} \tag{3.18}$$

Using the perturbation parameter notation, we can also write equations (3.15) and (3.16) as:

64

$$\lambda_i(\mu) = \bar{\lambda}_i + \varepsilon\,\lambda_{i_1} + \varepsilon^2\,\lambda_{i_2} \tag{3.19}$$

$$\phi_i(\mu) = \bar{\phi}_i + \varepsilon\,\phi_{i_1} + \varepsilon^2\,\phi_{i_2} \tag{3.20}$$

where:

$$\lambda_{i_1} = \sum_{m=1}^{R} \lambda_{i_m}^{I}\,\delta_m \quad : \quad \phi_{i_1} = \sum_{m=1}^{R} \phi_{i_m}^{I}\,\delta_m \tag{3.21}$$

$$\lambda_{i_2} = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} \lambda_{i_{mn}}^{II}\,\delta_m\,\delta_n \quad : \quad \phi_{i_2} = \frac{1}{2}\sum_{m=1}^{R}\sum_{n=1}^{R} \phi_{i_{mn}}^{II}\,\delta_m\,\delta_n \tag{3.22}$$

Substituting equations (3.13- 3.14) and (3.19- 3.20) in equation (3.1), the perturbed eigenvalue problem can be written as follows:

$$\left\{ \left([\bar{K}] + \varepsilon[K_1] + \varepsilon^2[K_2]\right) - \left(\bar{\lambda}_i + \varepsilon\,\lambda_{i_1} + \varepsilon^2\,\lambda_{i_2}\right) \right.$$
$$\left. \left([\bar{M}] + \varepsilon[K_1] + \varepsilon^2[M_2]\right)\right\} \left(\bar{\phi}_i + \varepsilon\,\phi_{i_1} + \varepsilon^2\,\phi_{i_2}\right) = 0 \tag{3.23}$$

Collecting the terms of the same order in the perturbation parameter, we obtain the following hierarchical equations:

$$O(\varepsilon^0): \quad \left\{[\bar{K}] - \bar{\lambda}_i\,[\bar{M}]\right\}\bar{\phi}_i = 0 \tag{3.24}$$

$$O(\varepsilon^1): \quad \left\{[\bar{K}] - \bar{\lambda}_i\,[\bar{M}]\right\}\phi_{i_1} =$$
$$= \left\{\bar{\lambda}_i\,[M_1] - [K_1]\right\}\bar{\phi}_i + \lambda_{i_1}\,[\bar{M}]\,\bar{\phi}_i \tag{3.25}$$

$$O(\varepsilon^2): \quad \left\{[\bar{K}] - \bar{\lambda}_i\,[\bar{M}]\right\}\phi_{i_2} =$$
$$= \left\{\bar{\lambda}_i\,[M_1] - [K_1]\right\}\phi_{i_1} + \left\{\bar{\lambda}_i\,[M_2] - [K_2]\right\}\bar{\phi}_i +$$
$$+ \lambda_{i_1}\,[M_1]\,\bar{\phi}_i + \lambda_{i_1}\,[\bar{M}]\,\phi_{i_1} + \lambda_{i_2}\,[\bar{M}]\,\bar{\phi}_i \tag{3.26}$$

The normalization condition (3.4) can also be expressed in terms of the perturbation expansions. Substituting equations (3.11- 3.14) in equation (3.4) leads to:

$$\left(\bar{\phi}_i + \varepsilon\,\phi_{i_1} + \varepsilon^2\,\phi_{i_2}\right)^T \left\{[\bar{M}] + \varepsilon[M_1] + \varepsilon^2[M_2]\right\} \left(\bar{\phi}_i + \varepsilon\,\phi_{i_1} + \varepsilon^2\,\phi_{i_2}\right) = 1 \tag{3.27}$$

The following equations are obtained after carrying out the products and collecting the terms of the same order in the perturbation parameter ε:

$$O(\varepsilon^0): \quad \bar{\phi}_i^T\,[\bar{M}]\,\bar{\phi}_i = 1 \tag{3.28}$$

$$O(\varepsilon^1): \quad 2\bar{\phi}_i^T\,[\bar{M}]\,\phi_{i_1} + \bar{\phi}_i^T\,[M_1]\,\bar{\phi}_i = 0 \tag{3.29}$$

$$O(\varepsilon^2): \quad 2\bar{\phi}_i^T\,[\bar{M}]\,\phi_{i_2} + 2\bar{\phi}_i^T\,[M_1]\,\phi_{i_1} + \bar{\phi}_{i_1}^T\,[\bar{M}]\,\phi_{i_1} + \bar{\phi}_i^T\,[M_2]\,\bar{\phi}_i = 0 \tag{3.30}$$

65

Zero-th Order Equations: The first equation (3.24) in the hierarchy of equations (3.24- 3.26) reproduces the original or unperturbed eigenproblem, whose eigenvalues and eigenvectors are $\bar{\lambda}_i$ and $\bar{\phi}_i$, respectively. Examining the corresponding $O - th$ order equation (3.28) we note that the original eigenvectors $\bar{\phi}i$ must be normalized with respect to the original mass matrix $[M]$.

First Order Equations: To obtain the first-order correction term for the eigenvalues we premultiply equation (3.25) by $\bar{\phi}_i^T$ and take advantage of the symmetry of the matrices $[K]$ and $[M]$. This leads to:

$$\lambda_{i_1} = \bar{\phi}_i^T \left\{ [K_1] + \bar{\lambda}_i [M_1] \right\} \bar{\phi}_i \tag{3.31}$$

Each first order perturbation term in this equation is defined in terms of a summation of first-order derivatives. If we substitute these summations and equate the coefficients of each variable δ_m . we obtain the following expression for the first-order eigenvalue derivative with respect to the $m - th$ variable:

$$\lambda_{i_m}^I = \bar{\phi}_i^T \left\{ \left[K_m^I \right] - \bar{\lambda}_i \left[M_m^I \right] \right\} \bar{\phi}_i \tag{3.32}$$

The corresponding first-order correction term for the eigenvectors, however, cannot be obtained directly from equation (3.25) since the coefficient matrix in the left hand side of equation (3.25) is singular. The solution ϕ_{i_1} would not be unique and it cannot be found without some previous reduction of the rank. We will use here an alternative approach to solve for the first-order perturbation terms ϕ_{i_1} . The method consists in expressing these terms as a linear combination of the eigenvectors of the unperturbed system, which constitute a valid set of basis vectors in a N-dimensional space and hence they can be used to represent any N-th order vector:

$$\phi_{i_1} = [\bar{\Phi}] \, \beta_{i_1} = \sum_{j=1}^{N} \beta_{i_{j_1}} \bar{\phi}_j \tag{3.33}$$

where $[\bar{\Phi}]$ is the modal matrix of the unperturbed problem, and the coefficient $\beta_{i_{j_1}}$ represents the projection of the perturbation term ϕ_{i_1} over the vector $\bar{\phi}_j$ of the basis. It is convenient to rewrite equation (3.33) as:

$$\phi_{i_1} = \gamma_{i_1} \bar{\phi}_i + \sum_{j=1, j \neq i}^{N} \beta_{i_{j_1}} \bar{\phi}_j \tag{3.34}$$

in which we have defined $\gamma_{i_1} = \beta_{i_{i_1}}$. If we substitute this expansion into the first-order equation (3.25). we obtain:

$$\left\{ [\bar{K}] - \bar{\lambda}_i [M] \right\} \sum_{j=1, j \neq i}^{N} \beta_{i_{j_1}} \bar{\phi}_j = \left\{ \bar{\lambda}_i [M_1] - [K_1] \right\} \bar{\phi}_i + \lambda_{i_1} [M] \bar{\phi}_i \tag{3.35}$$

We can now premultiply this equation by $\bar{\phi}_j^T$. with $j \neq i$ and make use of the orthogonality

66

relationships for the eigenvectors of the unperturbed system. It is straightforward to show that this leads to:

$$\beta_{ij1} = \left(\frac{1}{\lambda_j - \lambda_i} \right) \bar{\phi}_i^T \left\{ \lambda_i [M_1] - [K_1] \right\} \bar{\phi}_i \quad : \quad i \neq j \qquad (3.36)$$

Recalling that, according to equation (3.11), the first order terms $[M_1]$ and $[K_1]$ are defined as a summation of R first-order derivatives, we can equate the coefficients of each perturbation variable, and write equation (3.36) as:

$$\beta_{ij1} = \sum_{m=1}^{R} \beta_{ijm}^I \delta_m \quad : \quad i \neq j \qquad (3.37)$$

where:

$$\beta_{ijm}^I = \left(\frac{1}{\lambda_j - \lambda_i} \right) \bar{\phi}_i^T \left\{ \bar{\lambda}_i \left[M_m^I \right] - \left[K_m^I \right] \right\} \bar{\phi}_i \quad : \quad i \neq j \qquad (3.38)$$

Equation (3.29) will allow us to determine the remaining coefficient γ_{i1} in the definition of the first-order correction vector ϕ_{i1} in equation (3.34). If we premultiply equation (3.34) by $\bar{\phi}_i^T [M]$ we obtain:

$$\gamma_{i1} = \bar{\phi}_i^T [M] \phi_{i1} \qquad (3.39)$$

Substituting for the right hand side term from equation (3.29) in equation (3.39), we have:

$$\gamma_{i1} = -\frac{1}{2} \bar{\phi}_i^T [M_1] \bar{\phi}_i \qquad (3.40)$$

Finally, substituting for $[M_1]$ from equation (3.11) and equating coefficients we obtain:

$$\gamma_{i1} = \sum_{m=1}^{R} \gamma_{i_m}^I \delta_m \qquad (3.41)$$

where:

$$\gamma_{i_m}^I = -\frac{1}{2} \bar{\phi}_i^T \left[M_m^I \right] \bar{\phi}_i \qquad (3.42)$$

Substituting for β_{ij1} from equation (3.37) and for γ_{i1} from equation (3.41) in equation (3.34), ϕ_{i1} can be expressed as:

$$\phi_{i1} = \sum_{m=1}^{R} \left(\gamma_{i_m}^I \bar{\phi}_i + \sum_{j=1, j \neq i}^{N} \beta_{ijm}^I \bar{\phi}_j \right) \delta_m \qquad (3.43)$$

Comparing the second of equations (3.21) and equation (3.43) we conclude that the first-order derivatives of the $i - th$ eigenvector can be written as follows:

$$\phi^I_{i_m} = \gamma^I_{i_m} \bar{\phi}_i + \sum_{J=1, J \neq 1}^{N} \beta^I_{iJm} \bar{\phi}_J \quad ; \quad m = 1, 2, \ldots, R \tag{3.44}$$

Second Order Equations: To obtain the 2nd order correction terms for the eigenvalues we premultiply equation (3.26) by $\bar{\phi}^T_i$ and use again the symmetry of the matrices $[K]$ and $[M]$. It is straightforward to show that this leads to:

$$\lambda_{i_2} = \bar{\phi}^T_i \left\{ [K_1] - \bar{\lambda}_i [M_1] \right\} \phi_{i_1} + \bar{\phi}^T_i \left\{ [K_2] - \bar{\lambda}_i [M_2] \right\} \bar{\phi}_i -$$

$$- \lambda_{i_1} \bar{\phi}^T_i [M_1] \bar{\phi}_i - \lambda_{i_1} \bar{\phi}^T_i [\bar{M}] \phi_{i_1} \tag{3.45}$$

The first-order perturbation terms in this equation represent a summation of first-order derivatives while the second-order perturbation terms represent a double summation of second-order derivatives. The expressions for these summations are given by equations (3.11- 3.12) and (3.21-3.22). Equating the coefficients of each product of variables $\delta_m \delta_n$, we obtain the second-order eigenvalue derivative with respect to the $m, n - th$ variables as follows:

$$\lambda^{II}_{i_{mn}} = \bar{\phi}^T_i \left\{ \left[K^I_m \right] - \bar{\lambda}_i \left[M^I_m \right] \right\} \phi^I_{i_n} +$$

$$+ \bar{\phi}^T_i \left\{ \left[K^I_n \right] - \bar{\lambda}_i \left[M^I_n \right] \right\} \phi^I_{i_m} +$$

$$+ \bar{\phi}^T_i \left\{ \left[K^{II}_{mn} \right] - \bar{\lambda}_i \left[M^{II}_{mn} \right] \right\} \bar{\phi}_i - \lambda^I_{i_m} \bar{\phi}^T_i \left[M^I_n \right] \bar{\phi}_i -$$

$$- \lambda^I_{i_n} \bar{\phi}^T_i \left[M^I_m \right] \bar{\phi}_i - \lambda^I_{i_m} \bar{\phi}^T_i [\bar{M}] \phi^I_{i_n} - \lambda^I_{i_n} \bar{\phi}^T_i [\bar{M}] \phi^I_{i_m} \tag{3.46}$$

To obtain the expression of the second-order correction term for the eigenvectors, we will use the same approach than for the first-order case. Using the set of eigenvectors of the unperturbed system as a new basis for the N-dimensional space, it is possible to write the vector ϕ_{i_2} as follows:

$$\phi_{i_2} = [\bar{\Phi}] \, \beta_{i_2} = \sum_{J=1}^{N} \beta_{iJ2} \, \bar{\phi}_J \tag{3.47}$$

Here again this expression can be arranged more conveniently as:

$$\phi_{i_2} = \gamma_{i_2} \bar{\phi}_i + \sum_{J=1, J \neq 1}^{N} \beta_{iJ2} \, \bar{\phi}_J \tag{3.48}$$

Substituting the above expression into the second-order equation (3.26), we obtain:

$$\left\{ [\bar{K}] - \dot{\lambda}_i \, [\bar{M}] \right\} \sum_{j=1, j \neq i}^{N} \, \beta_{ij2} \, \dot{\phi}_j =$$

$$= \left\{ \bar{\lambda}_i \, [M_1] - [K_1] \right\} \, \phi_{i_1} + \left\{ \bar{\lambda}_i \, [M_2] - [K_2] \right\} \, \phi_i +$$

$$+ \lambda_{i_1} \, [M_1] \, \bar{\phi}_i + \lambda_{i_1} \, [\bar{M}] \, \phi{i_1} + \lambda_{i_2} \, [\bar{M}] \, \bar{\phi}_i \qquad (3.49)$$

In order to obtain the expression for β_{ij2}, we premultiply this equation by $\bar{\phi}_j^T$, with $j \neq i$. After applying the orthogonality relationships for the eigenvectors of the unperturbed system, we obtain:

$$\beta_{ij2} = \left(\frac{1}{\lambda_j - \lambda_i} \right) \left(\bar{\phi}_j^T \left\{ \bar{\lambda}_i \, [M_1] - [K_1] \right\} \, \phi_{i_1} + \right.$$

$$+ \bar{\phi}_j^T \left\{ \bar{\lambda}_i \, [\bar{M}] - [K_2] \right\} \, \bar{\phi}_i + \lambda_{i_1} \, \bar{\phi}_j^T \, [M_1] \, \bar{\phi}_i +$$

$$\left. \lambda_{i_1} \, \bar{\phi}_j^T \, [\bar{M}] \, \phi_{i_1} \right) \qquad ; \qquad i \neq j \qquad (3.50)$$

Each first and second order term in this equation is defined in terms of the single or double summations in equations (3.11- 3.12) and (3.21). Substituting these summations and equating the coefficients that multiply the variables $\delta_m \delta_n$, equation (3.50) reduces to:

$$\beta_{ij2} = \sum_{m=1}^{R} \sum_{n=1}^{R} \beta_{ijmn}^{II} \, \delta_m \, \delta_n \quad ; \quad i \neq j \qquad (3.51)$$

where:

$$\beta_{ijmn}^{II} = \left(\frac{1}{\lambda_j - \lambda_i} \right) \left(\bar{\phi}_j^T \left\{ \bar{\lambda}_i \, [M_m^I] - [K_m^I] \right\} \, \phi_{i_n}^I + \right.$$

$$+ \bar{\phi}_j^T \left\{ \bar{\lambda}_i \, [M_n^I] - [K_n^I] \right\} \, \phi_{i_m}^I +$$

$$+ \bar{\phi}_j^T \left\{ \bar{\lambda}_i \, [M_{mn}^{II}] - [K_{mn}^{II}] \right\} \, \bar{\phi}_i +$$

$$+ \lambda_{i_m}^I \, \bar{\phi}_j^T \, [M_n^I] \, \bar{\phi}_i + \lambda_{i_n}^I \, \bar{\phi}_j^T \, [M_m^I] \, \bar{\phi}_i +$$

69

$$+ \lambda_{i_m}^I \, \phi_j^T \, [M] \, \phi_{i_n}^I \, + \, \lambda_{i_n}^I \, \bar{\phi}_j^T \, [M] \, \phi_{i_m}^I \Big) \tag{3.52}$$

The term γ_{i_2} in equation (3.48) remains undefined. To obtain this term we need to utilize the 2nd order normalization condition. equation (3.30). First, we premultiply equation (3.48) by $\phi_i^T \, [M]$ to obtain:

$$\gamma_{i_2} = \bar{\phi}_i^T \, [M] \, \phi_{i_2} \tag{3.53}$$

The term in the right hand side of the above equation can be expressed in terms of known quantities using equation (3.30). This leads to:

$$\gamma_{i_2} = -\bar{\phi}_i^T \, [M_1] \, \phi_{i_1} \, - \, \frac{1}{2} \phi_{i_1}^T \, [M] \, \phi_{i_1} \, - \, \frac{1}{2} \bar{\phi}_i^T \, [M_2] \, \bar{\phi}_i \tag{3.54}$$

All the perturbation terms in this expression are defined in equations (3.11- 3.12) and (3.21) in terms of summations of the incremental variables δ_i . Substituting these expressions in equation (3.54) and re-arranging coefficients we obtain:

$$\gamma_{i_2} = \sum_{m=1}^{R} \sum_{n=1}^{R} \gamma_{imn}^{II} \, \delta_m \, \delta_n \quad : \quad i \neq j \tag{3.55}$$

in which:

$$\gamma_{imn}^{II} = -\bar{\phi}_i^T \, \left[M_m^I\right] \, \phi_{i_n}^I \, - \, \bar{\phi}_i^T \, \left[M_n^I\right] \, \phi_{i_m}^I \, - \, \phi_{i_m}^{IT} \, [M] \, \phi_{i_n}^I \, -$$

$$- \frac{1}{2} \bar{\phi}_i^T \, \left[M_{mn}^{II}\right] \, \bar{\phi}_i \tag{3.56}$$

To obtain the second-order derivatives of the eigenvectors we replace equations (3.51) and (3.55) in (3.48):

$$\phi_{i_2} = \sum_{m=1}^{R} \sum_{n=1}^{R} \left(\gamma_{imn}^{II} \, \bar{\phi}_i \, + \, \sum_{j=1,j\neq i}^{N} \beta_{ijmn}^{II} \, \bar{\phi}_j \right) \, \delta_m \, \delta_m \tag{3.57}$$

A comparison of equation (3.57) and the second equation (3.22) leads to the conclusion that the second-order derivatives of the $i - th$ eigenvector can be calculated as:

$$\phi_{imn}^{II} = \gamma_{imn}^{II} \, \bar{\phi}_i \, + \, \sum_{j=1,j\neq i}^{N} \beta_{ijmn}^{II} \, \bar{\phi}_j \tag{3.58}$$

70

3.2.1 Numerical Examples

The model used in this chapter to present numerical examples is the plane frame shown in Figure (2.16). The non-rigid joints are those at the ends of the three beams.

For the first example the joints (b),(d),(e) and (f) are assumed to be all rigid in one case (solid lines), and all flexible with the same fixity factor equal to 0.70 in the other case (dotted lines). We will examine the effect of introducing the same perturbations to the stiffness values of joints (a) and (c). Figure (3.1) shows the variation in percent in the lower eigenvalue of the system when the stiffness coefficients in the connections (a) and (c) are increased in 10% and 20%, and the stiffness coefficients of the remaining connections stay unchanged. It should be emphasized that the 10% and 20% perturbations are applied to the stiffness coefficients and not to the fixity factors. Figure (3.1) shows that the first eigenvalue is affected the most when the joints (a) and (c) have a fixity factor between 0.30 and 0.35. Within this interval, when the stiffness of these connections are increased by 10%, the eigenvalue increases in about 1.20% and when they are increased 20%, the eigenvalue changes in about 2.30%. The stiffness of the rest of the connections does not have a significant effect in the curves of variation. Changing the fixity factors of the remaining joints from 1.00 to 0.70 only slightly increase or decrease the amplitude of the variation. As it can be seen in the figure, the two curves intersect when the fixity factors of joints (a) and (c) is 0.38. Figures (3.2) and (3.3), respectively, show the variation in the magnitude of the second and third eigenvalues when the stiffness of the connections (a) and (c) are perturbed by 10% and 20%. The curves in Figure (3.2) show that the second eigenvalue is almost insensitive to the perturbations in the stiffness values. This fact was already noticed in the numerical examples of the previous chapter. For the case of the third eigenvalue the curves attain maximum values of 1.21% and 2.32%. The range of maximum sensitivity is between $\mu = 0.50$ and $\mu = 0.55$. In this case decreasing the stiffness of the remaining connections always provokes an increase in the curves of variation.

Similar curves describing the variation in the eigenvalues are presented in Figures (3.4-6), but this time the perturbations are introduced in the stiffness values of joints (a),(b),(c) and (d) of the frame. Two cases are considered again: the remaining joints (e) and (f) are either regarded as rigid or flexible with the same fixity factor equal to 0.70. The variations in the first and third eigenvalue are increased with respect to the previous example. The maximum variations are now 2.20% and 4.10% for the first eigenvalue and 2.40% and 4.50% for the third eigenvalue. The behavior of the second eigenvalue presented in Figure (3.5) differs notably with respect to the previous example. Not only the sensitivity of the eigenvalue increased but the shape of the curves present maxima for very low values of the fixity factors. Moreover, the value of the fixity factors of the remaining connections has a more pronounced effect in the variation curves.

The eigenvalues of the frame are nonlinear functions of the stiffness of the connections. For a structure with R flexible connections, a particular eigenvalue λ_i of the system can be represented as a surface in an $(R + 1)$-dimensional space. For the case in which only two connections are flexible it is possible to plot this curve. This type of analysis will permit us to quantify the relative importance of the perturbation of an individual connection. We will assume that the flexibility of the joints (a) and (c) can vary from the pinned case ($\mu = 0.00$) to the fixed case

71

($\mu = 1.00$) while the rest of the joints are rigid. The variation in the first three eigenvalues of the frame can be represented by three surfaces in Figures (3.7), (3.9) and (3.11). In these figures the eigenvalues were normalized by dividing their values by the corresponding eigenvalue associated with the rigid joints structure. The figures show that the sensitivity of the eigenvalues depends on which stiffness connection is varied, that is, the curves are not symmetrical with respect to a vertical plane at 45° with respect to the horizontal axes. These characteristics can be appreciated better using contour lines of the surfaces. They are shown in Figures (3.8), (3.10) and (3.12) for the first, second and third eigenvalue, respectively. It is noted that the dotted-line trayectories representing states of the system defined by equal fixity factors for joints (a) and (c) do not necessarily coincide the trayectories of maximum variation. For instance, Figure (3.8) shows that the gradient vector evaluated at fixity factors (0.40, 0.40) forms an angle with respect to the 45° line.

Figures (3.13-15) show the normalized components of the gradient vector along the μ_a and μ_c axes evaluated at $\mu_a = \mu_c$ for the first three eigenvalues. The gradient vector is defined as:

$$\nabla \lambda_i = \frac{\partial \lambda_i}{\partial \mu_a} e_a + \frac{\partial \lambda_i}{\partial \mu_b} e_b \tag{3.59}$$

where the vector e_m represents the unit vector along the μ_m axis and the derivatives were calculated using the expression (3.32) developed in this chapter.

Figure (3.13) shows the components of the gradient vector for the first eigenvalue as a function of the fixity factors. The graph reveals that perturbing the joint (a) has a more pronounced effect than perturbing the joint (c). Figure (3.14) shows that the relative contribution of joints (a) and (c) to the second eigenvalue, measured in terms of the gradient, depends on their fixity factor. For a fixity factor less than 0.85, joint (c) is predominant and, on the contrary, for values greater than 0.85 joint (a) is dominant. In the case of the third eigenvalue, Figure (3.15) shows that the third eigenvalue is almost completely insensitive to perturbations to the stiffness of joint (a).

We will examine next the effect on the components of the gradient vector of varying the stiffness of joints (a),(b),(c), and (d) from an initial state defined by equal values of the fixity factors. The gradient vector is now defined as:

$$\nabla \lambda_i = \frac{\partial \lambda_i}{\partial \mu_a} e_a + \frac{\partial \lambda_i}{\partial \mu_b} e_b + \frac{\partial \lambda_i}{\partial \mu_c} e_c + \frac{\partial \lambda_i}{\partial \mu_d} e_d \tag{3.60}$$

In Figures (3.16-18) are represented the normalized components of the gradient vector for the lower three eigenvalues. It is seen from Figure (3.16) that the joints (a) and (b) have a stronger influence on the first eigenvalue, although the difference is not very significant. In the case of the second eigenvalue considered in Figure (3.17), the influence of joint (c) is dominant except for higher values of the fixity factors. For instance, according to Figure (3.17) if the original value of the fixity factors for all the joints at the first level is 0.40, the maximum variation for the second eigenvalue is attained altering the fixity factor of joint (c). The components of the gradient vector for the third eigenvalue are shown in Figure (3.18). Contrary to the previous cases, it is seen that the third eigenvalue is not affected by perturbations in the stiffness of joints (a) and (b).
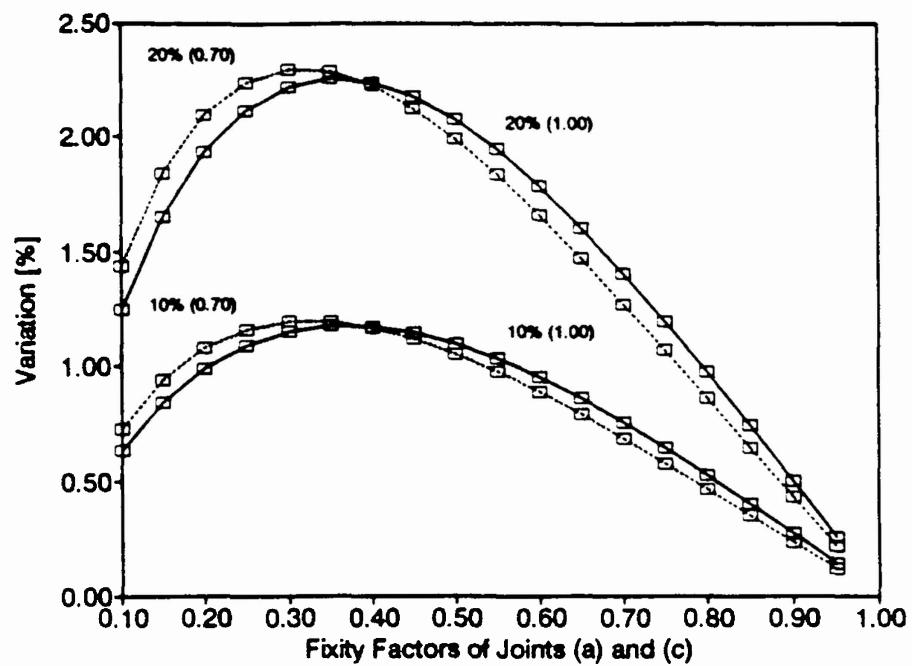
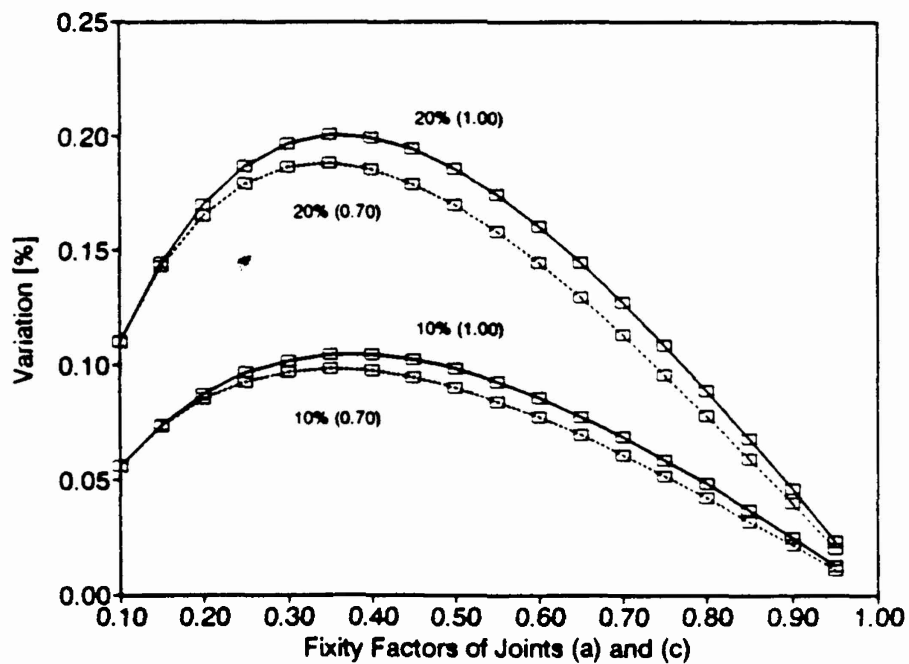Figure 3.1: Variation in First Eigenvalue due to Stiffness Perturbations of Joints (a) and (c).

Figure 3.2: Variation in Second Eigenvalue due to Stiffness Perturbations of Joints (a) and (c).
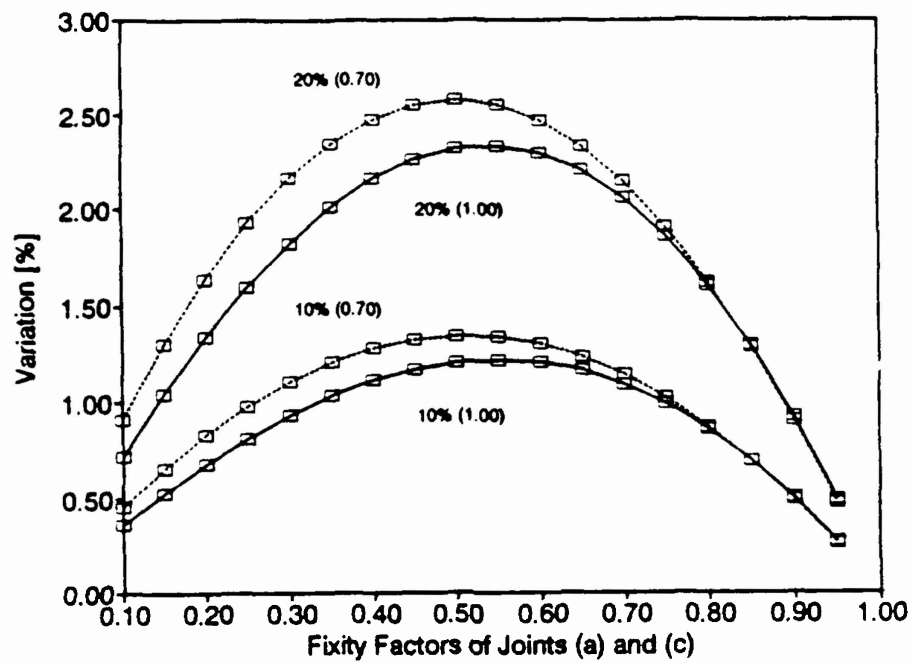
73

Figure 3.3: Variation in Third Eigenvalue due to Stiffness Perturbations of Joints (a) and (c).

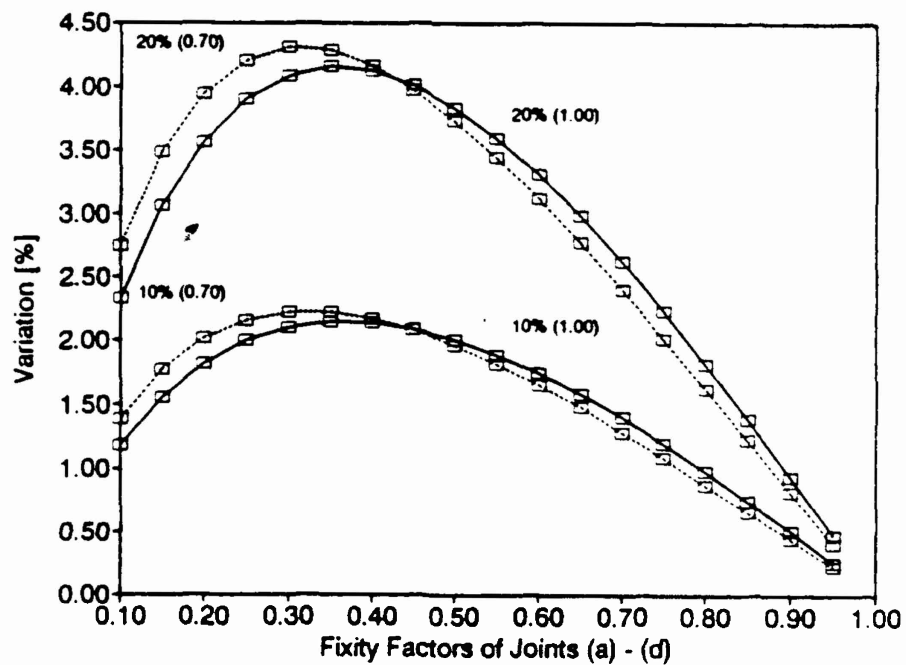Figure 3.4: Variation in First Eigenvalue due to Stiffness Perturbations of Joints (a)-(d).

Figure 3.5: Variation in Second Eigenvalue due to Stiffness Perturbations of Joints (a)-(d).
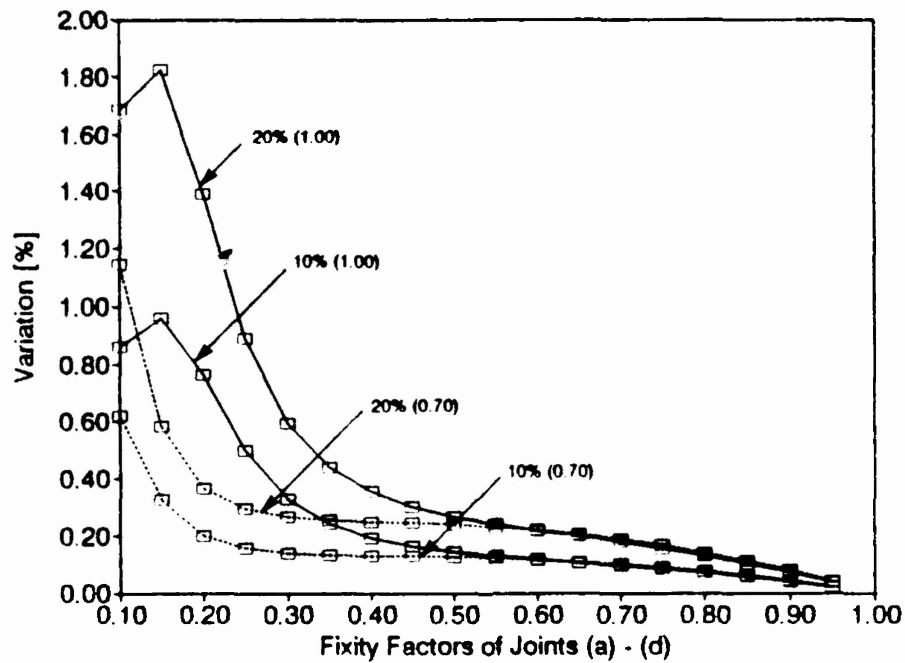
Figure 3.6: Variation in Third Eigenvalue due to Stiffness Perturbations of Joints (a)-(d).

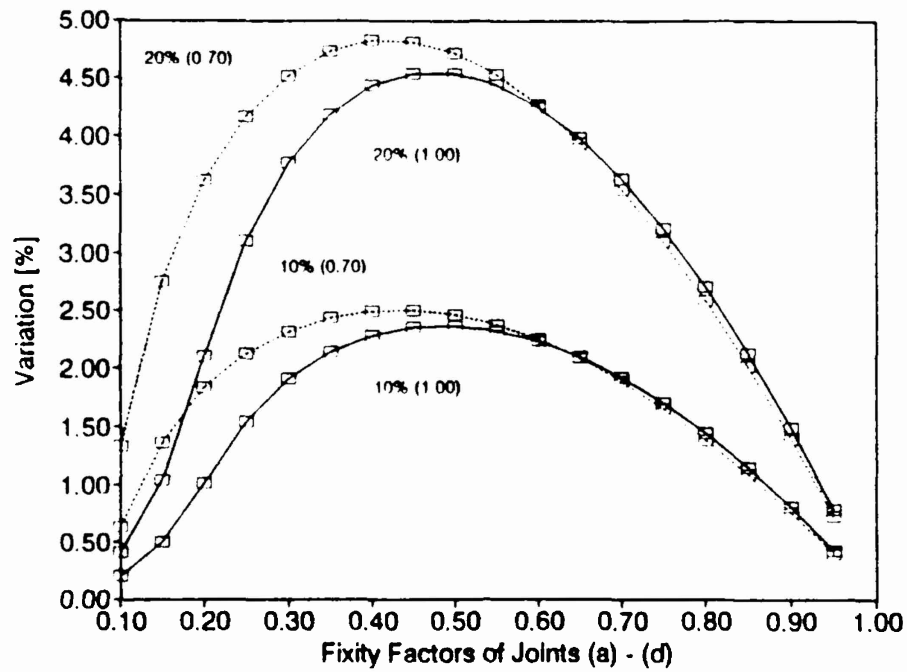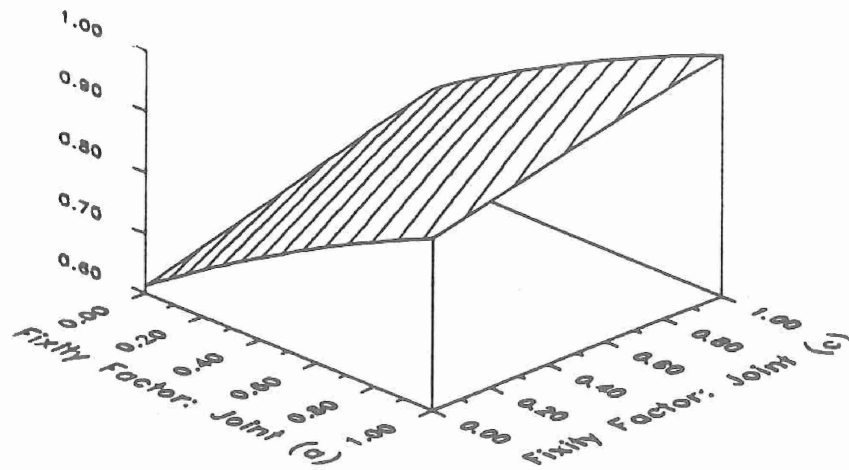Figure 3.7: Three-Dimensional Representation of First Eigenvalue.
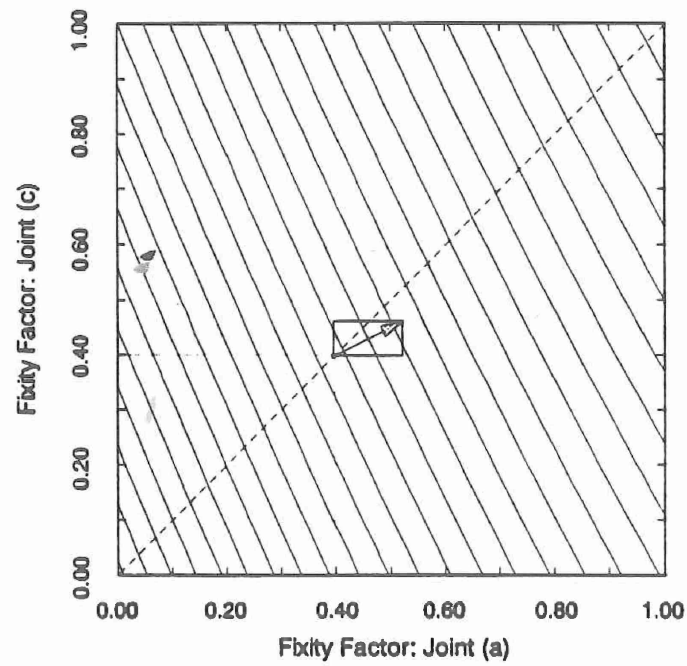
Figure 3.8: Contour Lines of First Eigenvalue Surface.
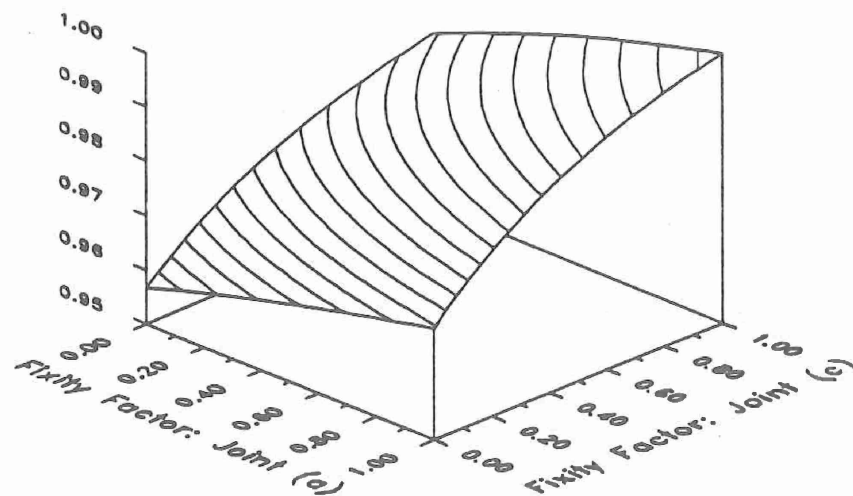
Figure 3.9: Three-Dimensional Representation of Second Eigenvalue.
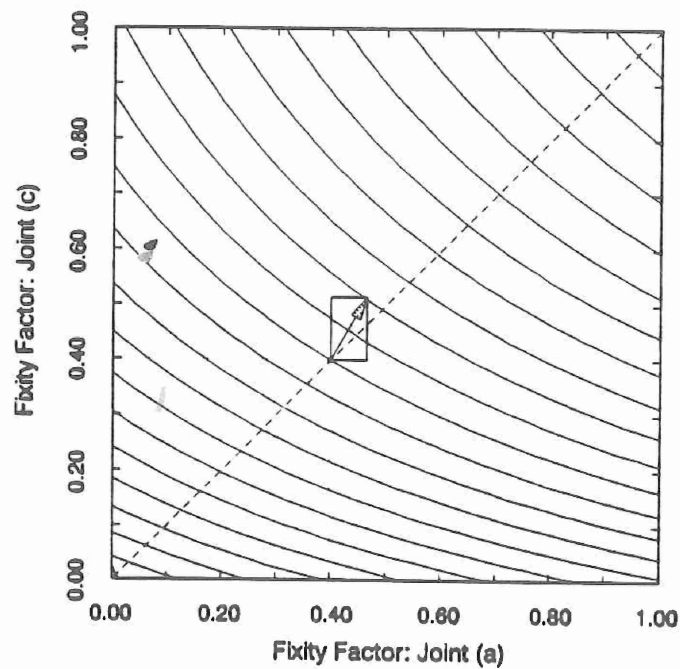
Figure 3.10: Contour Lines of Second Eigenvalue Surface.
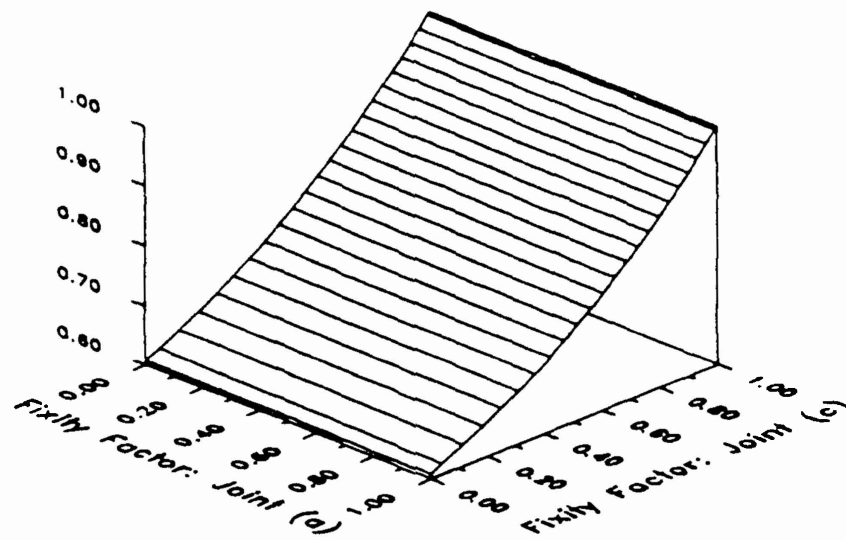
Figure 3.11: Three-Dimensional Representation of Third Eigenvalue.
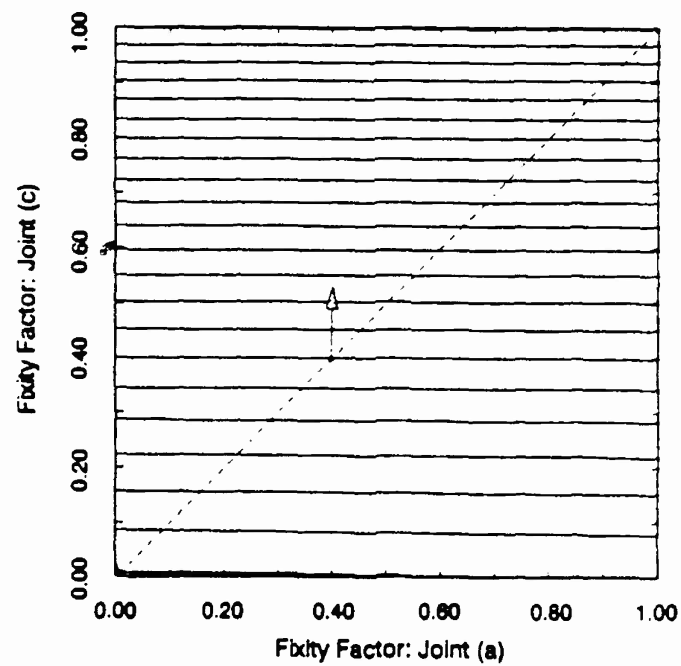
Figure 3.12: Contour Lines of Third Eigenvalue Surface.
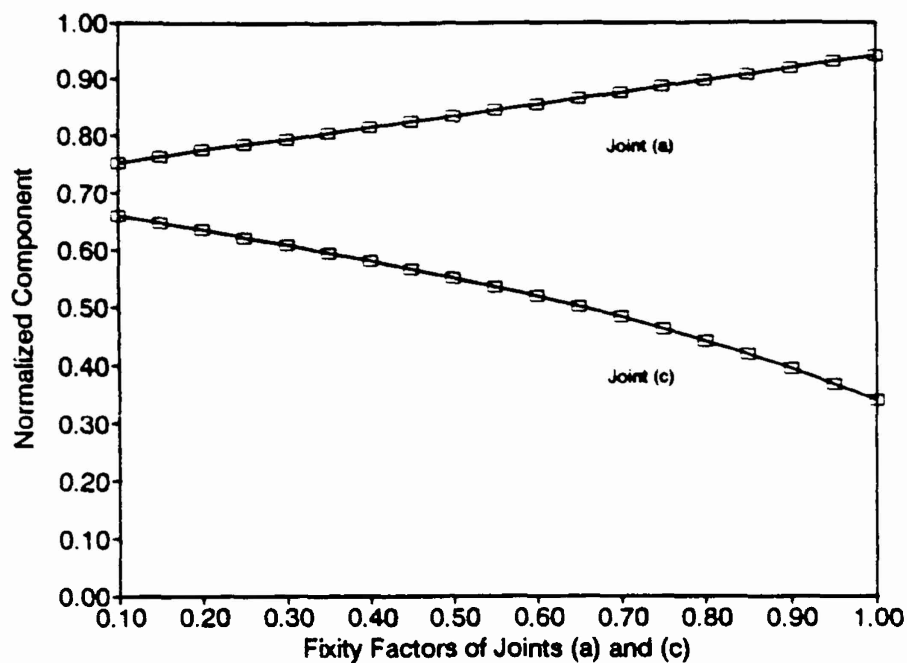
Figure 3.13: Gradient Components for First Eigenvalue as Function of Fixity Factors of Joints (a) and (c).
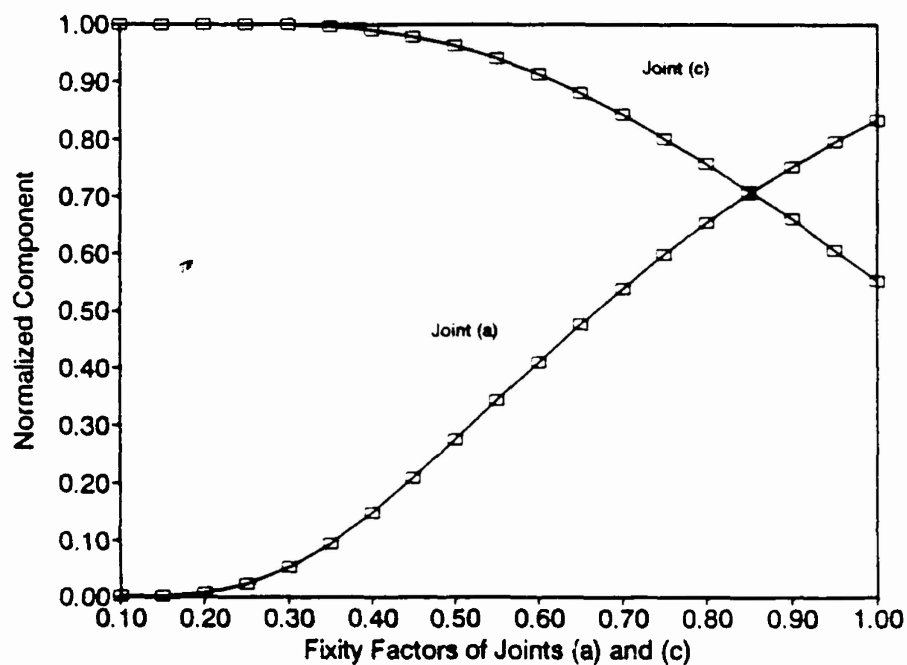
Figure 3.14: Gradient Components for Second Eigenvalue as Function of Fixity Factors of Joints (a) and (c).

79
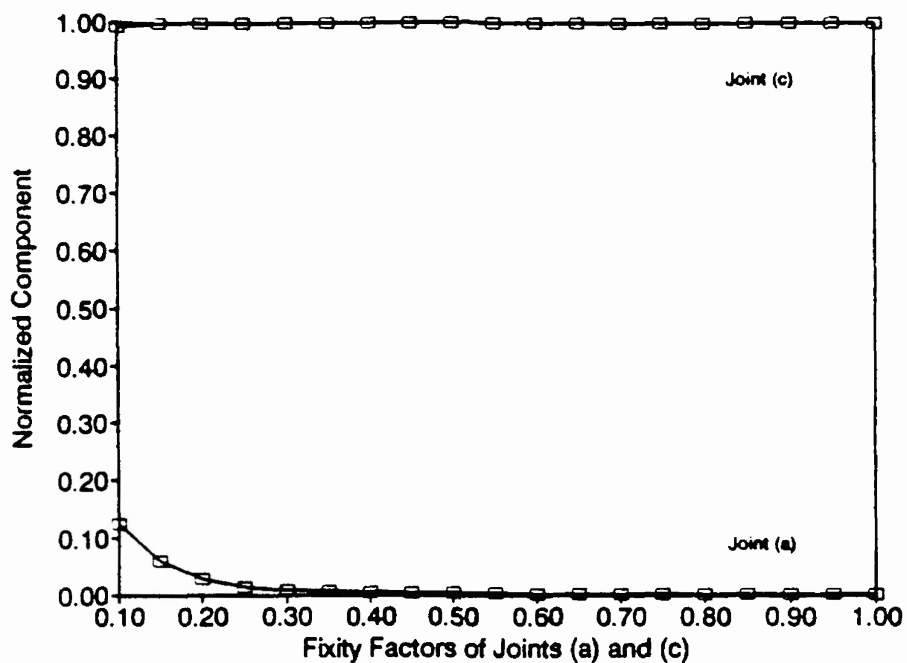
Figure 3.15: Gradient Components for Third Eigenvalue as Function of Fixity Factors of Joints (a) and (c).
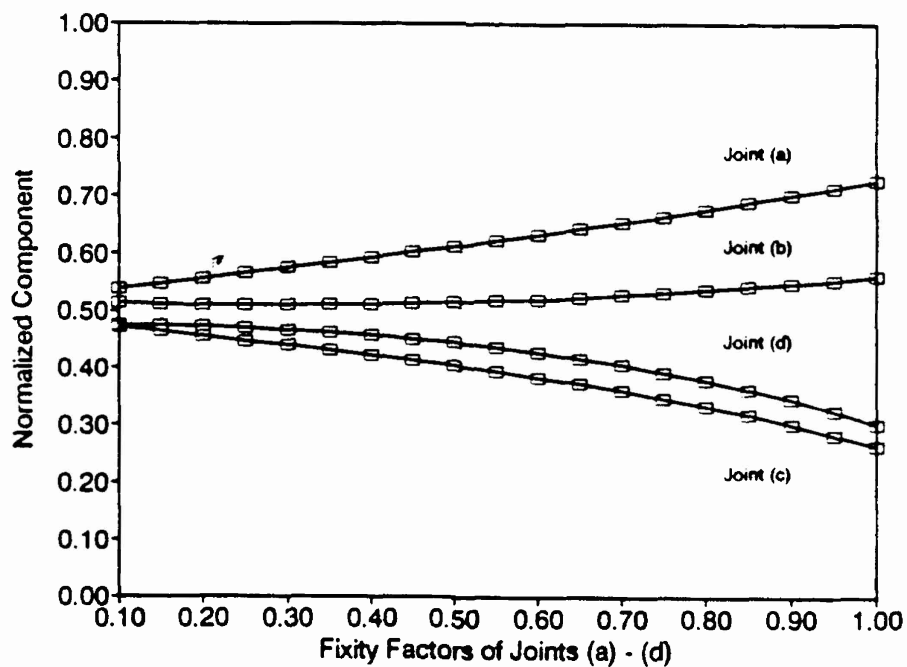
Figure 3.16: Gradient Components for First Eigenvalue as Function of Fixity Factors of Joints (a)-(d).
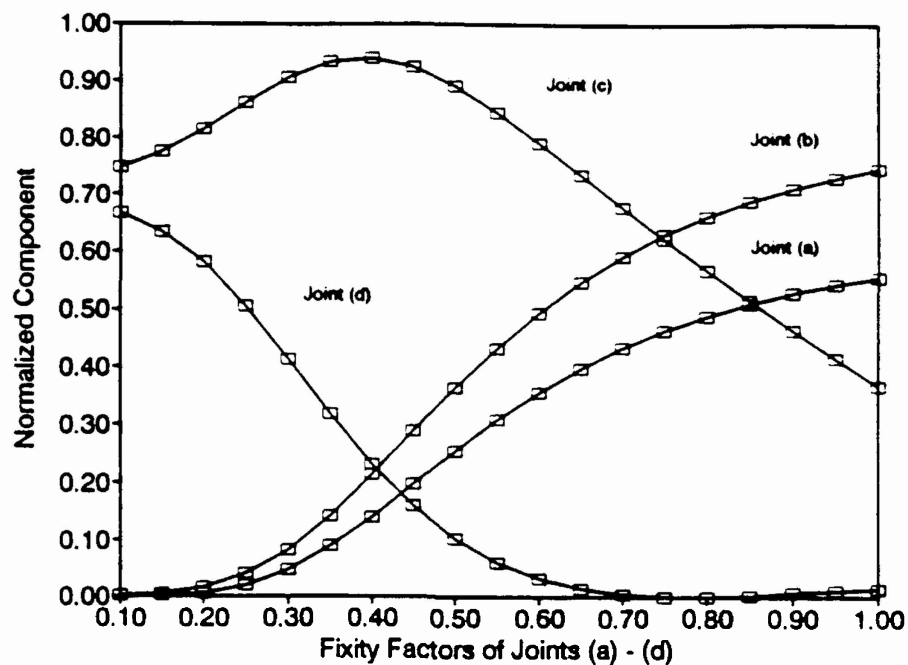
Figure 3.17: Gradient Components for Second Eigenvalue as Function of Fixity Factors of Joints (a)-(d).
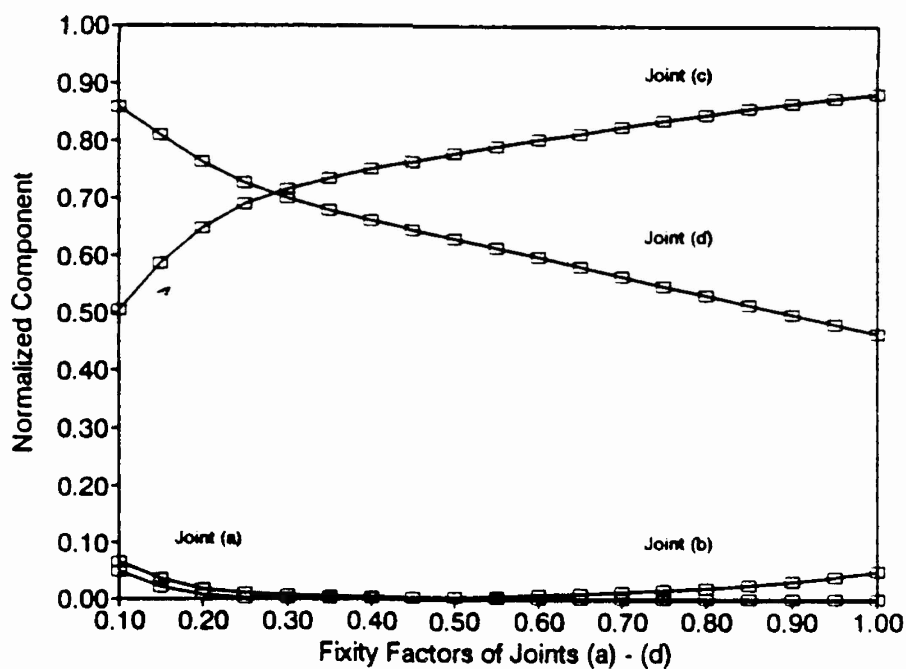
Figure 3.18: Gradient Components for Third Eigenvalue as Function of Fixity Factors of Joints (a)-(d).

81

Bibliography

[1] Kato, B., "Beam-to-Beam Column Connection Research in Japan", Journal of the Structural Division, ASCE, 108 (ST2), 343-360, 1982.

[2] Krawinkler, H. and Popov, E. P., "Seismic Behavior of Moment Resisting Connections and Joints", Journal of the Structural Division, ASCE, 108 (ST2), 373-391, 1982.

[3] Altman, W.G., Azizinamini, A., Bradburn, J.H. and Radziminski, J.B., "Moment-Rotation Characteristics of Semi-Rigid Steel Beam-Column Connections", Structural Research Studies, Department of Civil Engineering, University of South Carolina, Columbia, SC, June, 1982.

[4] Monforton, G.R. and Wu, T.S., "Matrix Analysis of Semi-Rigidly Connected Frames", Journal of the Structural Division, ASCE, 89 (ST6), 13-42, 1963.

[5] Romstad, K.M. and Subramanian, C.V., "Analysis of Frames with Partial Connection Rigidity", Journal of the Structural Division, ASCE, 96 (ST11), 2283-2300, 1970.

[6] Lightfoot, E. and LetMessurier, A.P., "Elastic Analysis of Frameworks with Elastic Connections", Journal of the Structural Division, ASCE, 100 (ST6), 1297-1309, 1974.

[7] Frye, M.J. and Morris, G.A., "Analysis of Flexibly Connected Steel Frames", Canadian Journal of Civil Engineering, 2, 280-291, 1975.

[8] Ang, K.M. and Morris, G.A., "Analysis of Three-Dimensional Frames with Flexible Beam-Column Connection", Canadian Journal of Civil Engineering, 11, 241-254, 1984.

[9] Arbabi, F., "Drift of Flexible Connection Frames", Computers and Structures, 15, 103-108, 1982.

[10] Chen, W.-F. and Lui, E.M., "Analysis and Behavior of Flexibly-Jointed Frames", Engineering Structures, 8, 107-118, 1986.

[11] Chen, W.-F. and Lui, E.M., "Effects of Joint Flexibility on the Behavior of Steel Frames", Computers and Structures, 26 (5), 719-732, 1987.

[12] Dhillon, B.S., and Abdel-Majid, S., "Interactive Analysis and Design of Flexibly Connected Frames", Computers and Structures, 36 (2), 189- 202, 1990.

[13] Livesley, R.K., Matrix Methods of Structural Analysis, 2nd Edition, Pergamon Press Ltd., Oxford, England, 1975.

[14] Lionberger, S. and Weaver, W., "Dynamic Response of Frames with Non-Rigid Connections", Journal of the Engineering Mechanics Division, ASCE, 95 (EM1), 95-114, 1969.

[15] Yaghmai, I. and Frohrib, D.A., "A Sensitivity Analysis of the Effects of Interconnection Joint Size, Flexibility and Inertia on the Natural Frequencies of Timoshenko Frames", Journal of Sound and Vibration, 75 (3), 329-346, 1981.

[16] Kawashima, S. and Fujimoto, T., "Vibration Analysis of Frames with Semi-Rigid Connections", Computers and Structures, 19, 85-92, 1984.

[17] Shi, G. and Atluri, S.N., "Static and Dynamic Analysis of Space Frames with Non-Linear Flexible Connections", International Journal for Numerical Methods in Engineering, 28, 2635-2650, 1989.

[18] Haug, E.J., Choi, K.K. and Komkov, V., Design Sensitivity Analysis of Structural Systems, Vol. 177, Mathematics in Science and Engineering, Academic Press, Orlando, FL, 1986.

[19] Adelman, H.M. and Haftka, R.T., "Sensitivity Analysis of Discrete Structural Systems", AIAA Journal, 24 (5), 823-832, 1986.

[20] Fox, R.L. and Kapoor, M.P., "Rates of Change of Eigenvalues and Eigenvectors", AIAA Journal, 6 (12), 2426-2429, 1968.

[21] Romstad, K.M., Hutchinson, J.R. and Runge, K.H., "Design Parameter Variation and Structural Response", International Journal for Numerical Methods in Engineering, 5, 337-349, 1973.

[22] Wilkinson, J.H., The Algebraic Eigenvalue Problem, Monographs on Numerical Analysis, Oxford University Press, London, England, 1965.

[23] Nelson, R.B., "Simplified Calculation of Eigenvector Derivatives", AIAA Journal, 14 (9), 1201-1205, 1976.

[24] Murthy, D.V. and Haftka, R.T., "Survey of Methods for Calculating Sensitivity of General Eigenproblems", Sensitivity Analysis in Engineering, NASA Conference Publication 2457, 177-195, 1986.

[25] Vanmarcke, E., Random Fields: Analysis and Synthesis, The MIT Press, Cambridge, MA, 1988.

[26] Srinivasan, S., Vasudevan, R., Introduction to Random Differential Equations and their Applications, Elsevier Publishing Company, Inc., New York, NY, 1971.

[27] Arnold, L., Stochastic Differential Equations: Theory and Applications, John Wiley & Sons, New York, NY, 1974.

83

[28] Schuss, Z., Theory and Application of Stochastic Differential Equations, John Wiley & Sons, New York, NY, 1980.

[29] Adomian, G., Stochastic Systems, Academic Press, New York, NY, 1983.

[30] Soong, T.T., Random Differential Equations in Science and Engineering, Academic Press, New York, NY, 1972.

[31] Vanmarcke, E., Shinozuka, M., Nakagiri, S., Schueller, G. and Grigoriu, M., "Random Fields and Stochastic Finite Element Methods", Structural Safety, 3, 143-166, 1986.

[32] Benaroya, H. and Rehak, M., "Finite Element Methods in Probabilistic Structural Analysis: A Selective Review", Applied Mechanics Review, 41 (5), 201-213, 1988.

[33] Vanmarcke, E. and Grigoriu, M., "Stochastic Finite Element Analysis of Simple Beams", Journal of the Engineering Mechanics Division, ASCE, 109 (EM5), 1203-1214, 1983.

[34] Der Kiureghian, A., "Finite Element Methods in Structural Safety Studies", Proc. ASCE Convention, Denver, Colorado, May 1985.

[35] Liu, W.K., Belytschko, T. and Mani, A., "Random Field Finite Elements", International Journal of Numerical Methods in Engineering, 23, 1831-1845, 1986.

[36] Liu, W.K., Belytschko, T. and Mani, A., "Probabilistic Finite Element Methods for Non-Linear Structural Dynamics", Computer Methods in Applied Mechanics and Engineering, 56, 61-81, 1986.

[37] Liu, W.K., Mani, A. and Belytschko, T., "Finite Element Methods in Probabilistic Mechanics", Probabilistic Engineering Mechanics, 2 (4), 201-213, 1987.

[38] Liu, W.K., Belytschko, T. and Mani, A., "Application of Probabilistic Finite Element Methods in Elastic Plastic Dynamics", Journal of Engineering for Industry, ASME, 109, 2-8, 1987.

[39] Liu, W.K., Besterfield, G. and Belytschko T., "Variational Formulation for Probabilistic Mechanics", Finite Elements Methods for Plates and Shells Structures: Formulation and Algorithms, eds. T.J.R. Hughes and E. Hinton, Pineridge Press, Swansea, England, 1986.

[40] Liu, W.K., Besterfield, G. and Belytschko T., "Transient Probabilistic Systems", Computer Methods in Applied Mechanics and Engineering, 67, 27-54, 1988.

[41] Liu, W.K., Besterfield, G.H. and Belytschko, T., "Variational Approach to Probabilistic Finite Elements", Journal of the Engineering Mechanics Division, ASCE, 114 (EM12), 2115-2133, 1988.

[42] Nakagiri, S., Hisada, T. and Toshimitsu, T., "Stochastic Time-History Analysis of Structural Vibration with Uncertain Damping", Probabilistic Structural Analysis, PVP ASME, 93, 109-120, 1984.

[43] Kareem, A. and Sun, W.-J., "Dynamic Response of Structures with Uncertain Damping", Engineering Structures, 12, 2-8, 1990.

[44] Hart, G.C., Uncertainty Analysis, Loads and Safety in Structural Analysis, Prentice-Hall Inc., Englewood Cliffs, NJ, 1982.

[45] Shinozuka, M. and Astill, C.J., "Random Eigenvalue Problems in Structural Analysis", AIAA Journal, 10 (4), 456-462. 1972.

[46] Shinozuka, M. and Dasgupta, G., "Stochastic Finite Element Methods in Dynamics", Proc. 3rd Conf. on Dynamic Response of Structures, Los Angeles, CA, 44-54, 1986.

[47] Shinozuka, M. and Deodatis, G., "Response Variability of Stochastic Finite Element Systems", Journal of the Engineering Mechanics Division, ASCE, 114 (EM3), 499-519, 1988.

[48] Yamazaki, F., Shinozuka, M. and Dasgupta, G., "Neumann Expansion for Stochastic Finite Element Analysis", Journal of the Engineering Mechanics Division, ASCE, 114 (EM8), 1335-1354, 1988.

[49] Boyce, W.E., "Random Eigenvalue Problems", Probabilistic Methods in Applied Mathematics, A.T. Barhucha-Reid ed., Academic Press, New York, NY, Vol. 1, 1-73, 1968.

[50] Collins, J.D. and Thomson, W.T., "The Eigenvalue Problem for Structural Systems with Statistical Properties", AIAA Journal, 7 (4), 642-648, 1969.

[51] Scheidt, J. and Purkert, G.M., Random Eigenvalue Problems, North-Holland, New York, NY, 1983.

[52] Hasselman, T.K. and Hart, G.C., "Modal Analysis of Random Structural Systems", Journal of the Engineering Mechanics Division, ASCE, 98 (EM3), 561-579, 1972.

[53] Hart, G.C., "Eigenvalue Uncertainty in Stressed Structures", Journal of the Engineering Mechanics Division, ASCE, 99 (EM3), 481-494, 1973.

[54] Boyce, W.E. and Goodwin, B.E., "Random Transverse Vibrations of Elastic Beams", SIAM Journal, 12, 613-629, 1964.

[55] Haines, C.W., "An Analysis of Stochastic Eigenvalue Problems", Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, NY, 1965.

[56] Iyengar, R.N. and Manohar, C.S., "Probability Distribution of the Eigenvalues of the Random String Equation", Journal of Applied Mechanics, ASME, 56, 202-207, 1989.

[57] Hurty, W.C., "Dynamic Analysis of Structural Systems by Component Mode Synthesis", AIAA Journal, 3 (4), 678-685, 1965.

[58] DERIVE User's Manual, Soft Warehouse, Inc., Honolulu, HI, 1991.

[59] Singh, M.P. and Chu, S.L., "Stochastic Consideration in Seismic Analysis of Structures", Earthquake Engineering and Structural Dynamics, 4, 295-307, 1976.

Appendix A

Explicit Form of Derivatives of the Element Matrices

This Appendix presents the explicit expressions of the derivatives of the stiffness and mass matrices of a flexural element with respect to the fixity factors μ_1 and μ_2. It is assumed that the connection eccentricity at the ends of the element is negligible. The derivative matrices are then obtained for eccentricity ratios f_1 and f_2 set equal to zero.

A.1 Mass Matrix

The mass matrix for a element with flexible connections of finite size is defined in equation (2.67) as the sum of two parts. The first component is constant with respect to the fixity factors. thus the only non zero contribution to the derivatives is due to the correction matrix defined in equation (2.75).

a) First derivative with respect to μ_1 :

$$
\left[\frac{\partial M}{\partial \mu_1}\right] = \frac{m\,L}{420}
\begin{bmatrix}
-156\,\eta_1(\mu_1,\mu_2) & & & \\
-22\,L\,\eta_2(\mu_1,\mu_2) & -4\,L^2\,\eta_5(\mu_1,\mu_2) & & \\
54\,\eta_3(\mu_1,\mu_2) & -13\,L\,\eta_6(\mu_1,\mu_2) & & \\
13\,L\,\eta_4(\mu_1,\mu_2) & 3\,L^2\,\eta_7(\mu_1,\mu_2) & & \\
& & \text{symmetric} & \\
& & -156\,\eta_8(\mu_1,\mu_2) & \\
& & 22\,L\,\eta_9(\mu_1,\mu_2) & -4\,L^2\,\eta_{10}(\mu_1,\mu_2)
\end{bmatrix}
\tag{A.1}
$$

b) First derivative with respect to μ_2 :

$$
\left[\frac{\partial M}{\partial \mu_2}\right] = \frac{m\,L}{420}
\begin{bmatrix}
-156\,\eta_8(\mu_1,\mu_2) & & & \\
-22\,L\,\eta_9(\mu_1,\mu_2) & -4\,L^2\,\eta_{10}(\mu_1,\mu_2) & & \\
54\,\eta_3(\mu_2,\mu_1) & -13\,L\,\eta_4(\mu_2,\mu_1) & & \\
13\,L\,\eta_6(\mu_1,\mu_2) & 3\,L^2\,\eta_7(\mu_2\mu_1) & & \\
& & \text{symmetric} & \\
& & -156\,\eta_1(\mu_2,\mu_1) & \\
& & 22\,L\,\eta_2(\mu_2,\mu_1) & -4\,L^2\,\eta_5(\mu_2,\mu_1)
\end{bmatrix}
\tag{A.2}
$$

86

c) Second derivative with respect to μ_1 :

$$\left[\frac{\partial^2 M}{\partial \mu_1^2}\right] = \frac{m\,L}{420} \begin{bmatrix} -156\,\xi_1(\mu_1,\mu_2) & & \\ -22\,L\,\xi_2(\mu_1,\mu_2) & -4\,L^2\,\xi_5(\mu_1,\mu_2) & \\ 54\,\xi_3(\mu_1,\mu_2) & -13\,L\,\xi_6(\mu_1,\mu_2) & \\ 13\,L\,\xi_4(\mu_1,\mu_2) & 3\,L^2\,\xi_7(\mu_1,\mu_2) & \end{bmatrix}$$

$$\begin{matrix} & symmetric & \\ -156\,\xi_8(\mu_1,\mu_2) & & \\ 22\,L\,\xi_9(\mu_1,\mu_2) & -4\,L^2\,\xi_{10}(\mu_1,\mu_2) \end{matrix} \Bigg] \qquad (A.3)$$

d) Second derivative with respect to μ_1 and μ_2 :

$$\left[\frac{\partial^2 M}{\partial \mu_1 \partial \mu_2}\right] = \frac{m\,L}{420} \begin{bmatrix} -156\,\xi_{11}(\mu_1,\mu_2) & & \\ -22\,L\,\xi_{12}(\mu_1,\mu_2) & -4\,L^2\,\xi_{15}(\mu_1,\mu_2) & \\ 54\,\xi_{13}(\mu_1,\mu_2) & -13\,L\,\xi_{14}(\mu_2,\mu_1) & \\ 13\,L\,\xi_{14}(\mu_1,\mu_2) & 3\,L^2\,\xi_{16}(\mu_1,\mu_2) & \end{bmatrix}$$

$$\begin{matrix} & symmetric & \\ -156\,\xi_{11}(\mu_2,\mu_1) & & \\ 22\,L\,\xi_{12}(\mu_2,\mu_1) & -4\,L^2\,\xi_{15}(\mu_2,\mu_1) \end{matrix} \Bigg] \qquad (A.4)$$

e) Second derivative with respect to μ_2 :

$$\left[\frac{\partial^2 M}{\partial \mu_2^2}\right] = \frac{m\,L}{420} \begin{bmatrix} -156\,\xi_{11}(\mu_2,\mu_1) & & \\ -22\,L\,\xi_{12}(\mu_2,\mu_1) & -4\,L^2\,\xi_{15}(\mu_2,\mu_1) & \\ 54\,\xi_3(\mu_2,\mu_1) & -13\,L\,\xi_4(\mu_2,\mu_1) & \\ 13\,L\,\xi_{14}(\mu_2,\mu_1) & 3\,L^2\,\xi_7(\mu_2,\mu_1) & \end{bmatrix}$$

$$\begin{matrix} & symmetric & \\ -156\,\xi_1(\mu_2,\mu_1) & & \\ 22\,L\,\xi_2(\mu_2,\mu_1) & -4\,L^2\,\xi_5(\mu_2,\mu_1) \end{matrix} \Bigg] \qquad (A.5)$$

where the following auxiliary functions were introduced for the definition of the first derivatives:

$$\eta_1(m,n) = -\left(50\,m\,n^3 - 72\,m\,n^2 - 216\,m\,n + 256\,m + 64\,n^3 - \right.$$
$$\left. - 192\,n^2 - 192\,n + 896\right) / \left(39\,R(m,n)^3\right) \qquad (A.6)$$

$$\eta_2(m,n) = (-2)\left(16\,m\,n^3 + 20\,m\,n^2 - 232\,m\,n + 256\,m + 64\,n^2 - \right.$$
$$\left. - 320\,n + 448\right) / \left(11\,R(m,n)^3\right) \qquad (A.7)$$

$$\eta_3(m,n) = \left(5\,m\,n^3 + 144\,m\,n^2 + 12\,m\,n - 512\,m + 128\,n^3 - \right.$$
$$\left. - 36\,n^2 + 384\,n - 112\right) / \left(27\,R(m,n)^3\right) \qquad (A.8)$$

$$\eta_4(m,n) = 2\left(19\,m\,n^3 - 170\,m\,n^2 + 256\,m\,n + 128\,n^3 - \right.$$

87

$$- 316\, n^2 + 200 \Big) / \Big(13\, R(m,n)^3 \Big) \tag{A.9}$$

$$\eta_5(m,n) = - \Big(64\, m\, n^2 - 248\, m\, n + 256\, m \Big) / \Big(R(m,n)^3 \Big) \tag{A.10}$$

$$\eta_6(m,n) = 2 \Big(32\, m\, n^3 - 170\, m\, n^2 - 44\, m\, n + 512\, m + 128\, n^2 +$$
$$+ 200\, n - 784 \Big) / \Big(13\, R(m,n)^3 \Big) \tag{A.11}$$

$$\eta_7(m,n) = \Big(64\, m\, n^3 - 372\, m\, n^2 + 512\, m\, n + 256\, n^2 -$$
$$- 496\, n \Big) / \Big(3\, R(m,n)^3 \Big) \tag{A.12}$$

$$\eta_8(m,n) = \Big(55\, m\, n^3 + 72\, m\, n^2 - 204\, m\, n - 256\, m - 64\, n^3 -$$
$$- 228\, n^2 + 192\, n + 784 \Big) / \Big(39\, R(m,n)^3 \Big) \tag{A.13}$$

$$\eta_9(m,n) = 2 \Big(43\, m\, n^3 - 20\, m\, n^2 - 128\, m\, n - 64\, n^3 -$$
$$- 52\, n^2 + 320\, n \Big) / \Big(11\, R(m,n)^3 \Big) \tag{A.14}$$

$$\eta_{10}(m,n) = \Big(31\, m\, n^3 - 64\, m\, n^2 - 64\, n^3 + 128\, n^2 \Big) / \Big(R(m,n)^3 \Big) \tag{A.15}$$

The following auxiliary functions were used to facilitate the definition of the second derivatives:

$$\xi_1(m,n) = - \Big(100\, m\, n^4 - 144\, m\, n^3 - 432\, m\, n^2 + 512\, m\, n + 192\, n^4 -$$
$$- 376\, n^3 - 864\, n^2 + 1824\, n + 1024 \Big) / \Big(39\, R(m,n)^4 \Big) \tag{A.16}$$

$$\xi_2(m,n) = (-2) \Big(32\, m\, n^4 + 40\, m\, n^3 - 464\, m\, n^2 + 512\, m\, n + 256\, n^3 -$$
$$- 880\, n^2 + 416\, n + 1024 \Big) / \Big(11\, R(m,n)^4 \Big) \tag{A.17}$$

$$\xi_3(m,n) = \Big(10\, m\, n^4 + 288\, m\, n^3 + 24\, m\, n^2 - 1024\, m\, n - 384\, n^4 -$$
$$- 88\, n^3 + 1728\, n^2 - 288\, n + 2048 \Big) / \Big(27\, R(m,n)^4 \Big) \tag{A.18}$$

$$\xi_4(m,n) = 2 \Big(38\, m\, n^4 - 340\, m\, n^3 + 512\, m\, n^2 + 384\, n^4 - 872\, n^3 -$$
$$- 80\, n^2 + 1024\, n \Big) / \Big(13\, R(m,n)^4 \Big) \tag{A.19}$$

$$\xi_5(m,n) = - \Big(128\, m\, n^3 - 496\, m\, n^2 + 512\, m\, n + 256\, n^2 -$$
$$- 992\, n + 1024 \Big) / \Big(R(m,n)^4 \Big) \tag{A.20}$$

$$\xi_6(m,n) = 2 \Big(64\, m\, n^4 - 340\, m\, n^3 - 88\, m\, n^2 + 1024\, m\, n + 512\, n^3 -$$
$$- 80\, n^2 - 2528\, n + 2048 \Big) / \Big(13\, R(m,n)^4 \Big) \tag{A.21}$$

$$\xi_7(m,n) = \Big(128\, m\, n^4 - 744\, m\, n^3 + 1024\, m\, n^2 + 1024\, n^3 -$$
$$- 2976\, n^2 + 2048 \Big) / \Big(3\, R(m,n)^4 \Big) \tag{A.22}$$

$$\xi_8(m,n) = \left(110\,m\,n^4 + 144\,m\,n^3 - 408\,m\,n^2 - 512\,m\,n - 192\,n^4 - \right.$$
$$\left. - 464\,n^3 + 864\,n^2 + 1536\,n - 1024\right) / \left(39\,R(m,n)^4\right) \tag{A.23}$$

$$\xi_9(m,n) = 2\left(86\,m\,n^4 - 40\,m\,n^3 - 256\,m\,n^2 - 192\,n^4 + 16\,n^3 - \right.$$
$$\left. - 880\,n^2 - 512\,n\right) / \left(11\,R(m,n)^4\right) \tag{A.24}$$

$$\xi_{10}(m,n) = \left(62\,m\,n^4 - 128\,m\,n^3 - 192\,n^4 + 496\,n^3 - 256\,n^2\right) / \left(R(m,n)^4\right) \tag{A.25}$$

$$\xi_{11}(m,n) = \left(72\,m^2\,n^2 + 432\,m^2\,n - 768\,m^2 - 408\,m\,n^2 + 960\,m\,n - \right.$$
$$\left. - 1824\,m - 768\,n^2 + 1536\,n + 768\right) / \left(39\,R(m,n)^4\right) \tag{A.26}$$

$$\xi_{12}(m,n) = (-2)\left(20\,m^2\,n^2 - 464\,m^2\,n + 768\,m^2 + 256\,m\,n + \right.$$
$$\left. + 416\,m + 512\,n - 1280\right) / \left(11\,R(m,n)^4\right) \tag{A.27}$$

$$\xi_{13}(m,n) = \left(144\,m^2\,n^2 + 24\,m^2\,n - 1536\,m^2 + 24\,m\,n^2 + 1920\,m\,n - \right.$$
$$\left. - 288\,m - 1536\,n^2 - 288\,n + 1536\right) / \left(27\,R(m,n)^4\right) \tag{A.28}$$

$$\xi_{14}(m,n) = (-2)\left(170\,m^2\,n^2 - 512\,m^2\,n + 88\,m\,n^2 + 960\,m\,n - \right.$$
$$\left. - 1024\,m - 1536\,n^2 + 2528\,n - 800\right) / \left(13\,R(m,n)^4\right) \tag{A.29}$$

$$\xi_{15}(m,n) = -\left(64\,m^2\,n^2 - 496\,m^2\,n + 768\,m^2 + 512\,m\,n - 992\,m\right) / \left(R(m,n)^4\right) \tag{A.30}$$

$$\xi_{16}(m,n) = -\left(372\,m^2\,n^2 - 1024\,m^2\,n - 1024\,m\,n^2 + 3968\,m\,n - \right.$$
$$\left. - 2048\,m - 2048\,n + 1984\right) / \left(3\,R(m,n)^4\right) \tag{A.31}$$

in which R(m,n) is given by equation (2.54):

$$R(m,n) = 4 - m\,n \tag{A.32}$$

A.2 Stiffness Matrix

The stiffness matrix for a element with flexible connections of finite size is given in equation (2.67) as the sum of three parts. As in the previous case, the non zero contributions to the derivatives are due to the second and third correction terms:

a) First derivative with respect to μ_1 :

$$\left[\frac{\partial K}{\partial \mu_1}\right] = \frac{2\,E\,I}{L^3} \begin{bmatrix} -6\,\delta_1(\mu_1,\mu_2) & \\ -3\,L\,\delta_2(\mu_1,\mu_2) & -2\,L^2\,\delta_3(\mu_1,\mu_2) \\ 6\,\delta_1(\mu_1,\mu_2) & 3\,L\,\delta_2(\mu_1,\mu_2) \\ 3\,L\,\delta_2(\mu_1,\mu_2) & -L^2\,\delta_4(\mu_1,\mu_2) \end{bmatrix}$$

$$
\begin{array}{c}
\hspace{3cm} symmetric \\
-6\,\delta_1(\mu_1,\mu_2) \\
3\,L\,\delta_2(\mu_1,\mu_2) \quad -2\,L^2\,\delta_3(\mu_1,\mu_2)
\end{array}
\Bigg] \tag{A.33}
$$

b) First derivative with respect to μ_2 :

$$
\left[\frac{\partial K}{\partial \mu_2}\right] = \frac{2\,E\,I}{L^3}
\left[
\begin{array}{cc}
-6\,\delta_1(\mu_2,\mu_1) & \\
-3\,L\,\delta_3(\mu_2,\mu_1) & -2\,L^2\,\delta_6(\mu_2,\mu_1) \\
6\,\delta_1(\mu_2,\mu_1) & 3\,L\,\delta_3(\mu_2,\mu_1) \\
-3\,L\,\delta_2(\mu_2,\mu_1) & -\,L^2\,\delta_5(\mu_2,\mu_1)
\end{array}
\right.
$$

$$
\begin{array}{c}
\hspace{3cm} symmetric \\
-6\,\delta_1(\mu_2,\mu_1) \\
3\,L\,\delta_2(\mu_2,\mu_1) \quad -2\,L^2\,\delta_4(\mu_2,\mu_1)
\end{array}
\Bigg] \tag{A.34}
$$

c) Second derivative with respect to μ_1 :

$$
\left[\frac{\partial^2 K}{\partial \mu_1^2}\right] = \frac{2\,E\,I}{L^3}
\left[
\begin{array}{cc}
-6\,\rho_1(\mu_1,\mu_2) & \\
-3\,L\,\rho_2(\mu_1,\mu_2) & -2\,L^2\,\rho_4(\mu_1,\mu_2) \\
6\,\rho_1(\mu_1,\mu_2) & 3\,L\,\rho_2(\mu_1,\mu_2) \\
3\,L\,\rho_3(\mu_1,\mu_2) & -\,L^2\,\rho_5(\mu_1,\mu_2)
\end{array}
\right.
$$

$$
\begin{array}{c}
\hspace{3cm} symmetric \\
-6\,\rho_1(\mu_1,\mu_2) \\
3\,L\,\rho_3(\mu_1,\mu_2) \quad -2\,L^2\,\rho_6(\mu_1,\mu_2)
\end{array}
\Bigg] \tag{A.35}
$$

d) Second derivative with respect to μ_1 and μ_2 :

$$
\left[\frac{\partial^2 K}{\partial \mu_1 \partial \mu_2}\right] = \frac{2\,E\,I}{L^3}
\left[
\begin{array}{cc}
-6\,\rho_7(\mu_1,\mu_2) & \\
-3\,L\,\rho_8(\mu_1,\mu_2) & -2\,L^2\,\rho_4(\mu_2,\mu_1) \\
6\,\rho_7(\mu_1,\mu_2) & 3\,L\,\rho_8(\mu_1,\mu_2) \\
3\,L\,\rho_8(\mu_2,\mu_1) & -\,L^2\,\rho_9(\mu_1,\mu_2)
\end{array}
\right.
$$

$$
\begin{array}{c}
\hspace{3cm} symmetric \\
-6\,\rho_7(\mu_1,\mu_2) \\
3\,L\,\rho_8(\mu_2,\mu_1) \quad -2\,L^2\,\rho_4(\mu_1,\mu_2)
\end{array}
\Bigg] \tag{A.36}
$$

e) Second derivative with respect to μ_2 :

$$
\left[\frac{\partial^2 K}{\partial \mu_2^2}\right] = \frac{2\,E\,I}{L^3}
\left[
\begin{array}{cc}
-6\,\rho_1(\mu_2,\mu_1) & \\
-3\,L\,\rho_3(\mu_2,\mu_1) & -2\,L^2\,\rho_6(\mu_2,\mu_1) \\
6\,\rho_1(\mu_2,\mu_1) & 3\,L\,\rho_3(\mu_2,\mu_1) \\
-3\,L\,\rho_2(\mu_2,\mu_1) & -\,L^2\,\rho_5(\mu_2,\mu_1)
\end{array}
\right.
$$

$$
\begin{array}{c}
\hspace{3cm} symmetric \\
-6\,\rho_1(\mu_2,\mu_1) \\
3\,L\,\rho_2(\mu_2,\mu_1) \quad -2\,L^2\,\rho_4(\mu_2,\mu_1)
\end{array}
\Bigg] \tag{A.37}
$$

where the following auxiliary functions were introduced for the definition of the first derivatives:

$$\delta_1(m,n) \quad = \quad -\left(n^2 + 4n + 4\right)/R(m,n)^2 \tag{A.38}$$

$$\delta_2(m,n) \quad = \quad (-4)(n+2)/R(m,n)^2 \tag{A.39}$$

$$\delta_3(m,n) \quad = \quad (-4)\left(n^2 + 2n\right)/R(m,n)^2 \tag{A.40}$$

$$\delta_4(m,n) \quad = \quad (-12)/R(m,n)^2 \tag{A.41}$$

$$\delta_5(m,n) \quad = \quad (-12)\,n/R(m,n)^2 \tag{A.42}$$

$$\delta_6(m,n) \quad = \quad (-3)\,n^2/R(m,n)^2 \tag{A.43}$$

The following auxiliary functions were used to facilitate the definition of the second derivatives:

$$\rho_1(m,n) \quad = \quad (-2)\,n\left(n^2 + 4n + 4\right)/R(m,n)^3 \tag{A.44}$$

$$\rho_2(m,n) \quad = \quad (-8)\,n\,(n+2)/R(m,n)^3 \tag{A.45}$$

$$\rho_3(m,n) \quad = \quad (-4)\left(n^3 + 2n^2\right)/R(m,n)^3 \tag{A.46}$$

$$\rho_4(m,n) \quad = \quad (-24)\,n/R(m,n)^3 \tag{A.47}$$

$$\rho_5(m,n) \quad = \quad (-24)\,n^2/R(m,n)^3 \tag{A.48}$$

$$\rho_6(m,n) \quad = \quad (-6)\,n^3/R(m,n)^3 \tag{A.49}$$

$$\rho_7(m,n) \quad = \quad (-4)\,(mn + 2mn + 2n + 4)/R(m,n)^3 \tag{A.50}$$

$$\rho_8(m,n) \quad = \quad (-4)\,(mn + 4m + 4)/R(m,n)^3 \tag{A.51}$$

$$\rho_9(m,n) \quad = \quad (-12)\,(mn + 4)/R(m,n)^3 \tag{A.52}$$

91